FUNCTION SERIES WITH MULTIFRACTAL VARIATIONS

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ABSTRACT. In this paper, we study three classes of multifractal function series using a work achieved for a new class of measures defined in [3].

The originality of these function series consists in the fact that the sizes of the jumps (or of the amplitudes of the pulses) depend on the location of the jump points and on a measure μ. In particular, there may be a strong heterogeneity in the distribution of the size of the jumps. These function series f are defined by

$$f(x) = \sum_{j \geq 1} \frac{1}{j^2} \sum_{b=0}^{b^j-1} \mu([kb^{-j}, (k+1)b^{-j}))\psi_{j,k}(x),$$

where $\psi_{j,k}$ is a contracted and dilated version of a single function ψ. This function ψ will either be a wavelet, a pulse, or a piecewise linear function.

We show that under suitable conditions on the measure μ, the multifractal spectrum of f can be computed. For large classes of measures, the spectrum is linear between 0 and a critical value h_c, and if $h \geq h_c$, f and μ share the same spectrum. This untypical shape is the result of the combination of the multifractal measure μ with the rapid variations or discontinuities of the functions $\psi_{j,k}$.

1. INTRODUCTION

In various scientific areas, many phenomena exhibit a wild varying regularity. They may have a multifractal structure, some of them may be discontinuous. In many works such objects are modeled by samples of processes which are continuous but with a low regularity. However, some of the underlying phenomena mentioned above are intrinsically discontinuous (for instance the Internet traffic [14]), and cannot be efficiently represented by samples of regular processes.

The multifractal functions and processes which do not belong to $C^\varepsilon(\mathbf{R})$ for some $\varepsilon > 0$ (i.e. which do not have a positive minimal uniform regularity) are not so frequent: Lévy processes (studied in [11]) with positive Lévy measure, Riemann functions [19], "Davenport Series" [7, 10] and a few other well-known functions [9]. All these examples can be viewed as (infinite) sums of discontinuous functions f_j. Their specificity is that the points where f_j has a discontinuity of a given size are homogeneously spatially distributed. This structure implies that their multifractal spectrum is linearly increasing (or composed of a linear part and of an isolated point).

This article contains other construction types. We allow the repartition of the locations of the jump points that have a given size to be highly heterogeneous. Our work consists in studying of the natural extensions of the construction of the measures detailed in [3] to the function's context. In this previous work [3], the following class of measures ν, defined as sums of measure-weighted Dirac masses, is introduced and studied: If μ is a positive Borel

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measure, ν has the following form which combines additive and multiplicative chaos

$$\nu = \sum_{j\geq 1} \frac{1}{j^2} \sum_{k=0}^{b^j-1} \mu([kb^{-j},(k+1)b^{-j}))\,\delta_{kb^{-j}}, \tag{1.1}$$

where b is an integer ≥ 2.

We are interested in the local behavior of such objects. The local regularity of a function or a measure at a point x is usually described by an Hölder exponent $h(x)$. When working with a positive Borel measure μ, this (lower) Hölder exponent is defined by

$$h_\mu(x) = \liminf_{r\to 0^+} \frac{\log \mu(B(x,r))}{\log r}.$$

When dealing with a function f, the local regularity of f at x is often measured by the pointwise Hölder exponent $h_f(x) = \sup\{h : f \in C^h_x\}$, where f is said to belong to the space C^h_x if one can find a polynomial P of degree smaller than $[h]$ and a constant C such that

$$\forall\, y \text{ close enough to } x,\ |f(y) - P(y-x)| \leq C|y-x|^h.$$

In both cases, the multifractal analysis consists in studying the size of the level sets $E^h = \{x : h(x) = h\}$ of these Hölder exponents. Then one computes the multifractal spectrum of the object, which is defined by

$$h \to d(h) = \dim(E^h),$$

where $\dim E$ denotes the Hausdorff dimension of the set E.

It is shown in [3] that under reasonable assumptions on μ (which are fulfilled by most of the usual classes of statistically self-similar measures, such as quasi-Bernoulli measures, Mandelbrot cascades, see [4]), the multifractal spectrum of ν can be computed. Moreover, in these cases, the measure ν satisfies a multifractal formalism for measures. This notion is important to fully understand the following theorems.

A multifractal formalism is a formula which relates, via a Legendre transform, the multifractal spectrum d_μ to a scaling function associated with μ (see [5], [18]). When working with measures, a possible definition for the scaling function [5] is : For any $q \in \mathbf{R}$,

$$\tau_\mu(q) = \liminf_{j\to+\infty} -\frac{1}{j}\log_b \left({\sum_{0\leq k\leq b^j-1}}^{*} \mu^q([kb^{-j},(k+1)b^{-j})) \right), \tag{1.2}$$

where $\sum^*$ means that the sum is taken over the integers k such that $\mu([kb^{-j},(k+1)b^{-j})) \neq 0$. The multifractal formalism is said to hold for μ at exponent h when the multifractal spectrum coincides with the Legendre transform of τ_μ at h:

$$d_\mu(h) = \tau_\mu^*(h) = \inf_{q\in\mathbf{R}} (qh - \tau_\mu(h)). \tag{1.3}$$

A large part of the work in multifractal analysis of measures is devoted to confirm or to refute the validity of this formula.

The multifractal analysis of functions constitutes a companion domain of the multifractal analysis of measures. Inspired by the construction of the measures ν, it is natural to build functions with a structure comparable with the one of ν. One hopes that the multifractal spectrum (for functions) of these functions is of interest.

Let μ be a positive Borel measure with a support equal to $[0,1]$.

A first interesting set of functions could be called positive lacunary heterogeneous Davenport series, in contrast with the Davenport series studied in [10]. Their multifractal spectrum can be deduced from the work achieved for the measure ν.

Let ϕ be the function $x \mapsto \phi(x) = x$ if $x \in [0, 1]$, and $\phi(x) = 0$ elsewhere. We study in Section 3 the function series defined by

$$(1.4)\quad f(x) = \sum_{j=1}^{+\infty} f_j(x) \text{ where } f_j(x) = \frac{1}{j^2} \sum_{k=0}^{b^j-1} \mu([(k+1)b^{-j}, (k+2)b^{-j}))\phi(b^j x - k).$$

For technical reasons, we choose the slope of each small linear part of f_j to be equal to the mass of the b-adic interval $I^+_{j,k} = [(k + 1)b^{-j}, (k + 2)b^{-j})$ instead of $I_{j,k} = [kb^{-j}, (k + 1)b^{-j})$.

The main, and exciting, difference with all the previous deterministic examples (Davenport or Riemann series) is that at each scale j, the function $x \mapsto f_j(x)$ is not periodic at all. There is a strong heterogeneity in the several slopes of this piecewise linear function f_j.

In order to be able to compute the multifractal spectrum of f, some technical conditions are required on the measure μ. These conditions **C1**, **C2** and **C3** (the same as the ones needed to compute the multifractal spectrum of ν (1.1)) are justified and explained in large details in [3]. We recall their consequences in Section 2.2.

Theorem 1.1. *Let μ be a positive Borel measure whose support is $[0, 1]$, and let us assume that **C1** holds for μ. Let $q_c = \inf\{q \in \mathbf{R} : \tau_\mu(q) = 0\}$ and $h_c = \tau'_\mu(q_c^-)$.*
*(1) If $h_c > 0$ and **C2**(h_c) holds, for any $h \in [0, h_c]$, $d_f(h) = q_c h$.*
*(2) If $h \geq h_c$ and **C3**(h) holds, then $d_f(h) = d_\mu(h) = \tau^*_\mu(h)$.*

Theorem 1.1 provides large classes of function series which (to our knowledge are the only ones that) can have an infinite number of jump discontinuities and a strictly concave multifractal spectrum with a continuous and non-trivial decreasing part.

We also study the case where the size of the jump points of f_j at kb^{-j} is modified with the help of two positive parameters γ and σ into $b^{-j\gamma}\mu([(k+1)b^{-j}, (k+2)b^{-j}))^\sigma$. We indicate, without proof, how the multifractal spectrum of the corresponding series behaves after the introduction of the two parameters.

Let us now introduce the following class of sum of pulses. Assume that the support of the measure μ is $[0, 1]$. Let $D = \liminf_{j\to\infty} \frac{\log \inf_{0\leq k<b^j-1} \mu(I_{j,k})}{\log(b^{-j})}$. Let χ be any function of $C^{[D]+2}(\mathbf{R})$ such that supp $\chi \subset [-1, 1]$. This function shall be considered as an atomic pulse. Section 4 is devoted to the analysis of the function series g defined by

$$(1.5)\qquad g(x) = \sum_{j=1}^{+\infty} \frac{1}{j^2} \sum_{\substack{0\leq k\leq b^j-1 \\ k\neq 0 \mod b}} \mu([kb^{-j}, (k+1)b^{-j}))\chi\big((x - kb^{-j})b^{j\log j}\big).$$

The function $\chi\big((\cdot - kb^{-j})b^{j\log j}\big)$ is supported by $[kb^{-j} - b^{j\log j}, kb^{-j} + b^{j\log j}]$.

Let us denote by $\widetilde{\mathbf{C}}\mathbf{1}$ the following property:

$$(1.6)\qquad \exists\, \gamma > 0 \text{ such that for every b-adic subinterval } I \text{ of } [0, 1], \mu(I) \leq |I|^\gamma.$$

Theorem 1.2. *Let μ be a positive Borel measure with a support equal to $[0, 1]$, and assume that **C1** and $\widetilde{\mathbf{C}}\mathbf{1}$ hold for μ. Let $q_c = \inf\{q \in \mathbf{R} : \tau_\mu(q) = 0\}$ and $h_c = \tau'_\mu(q_c^-)$.*
*(1) If $h_c > 0$ and **C2**(h_c) holds, for any $h \in [0, h_c)$, $d_g(h) = q_c h$.*

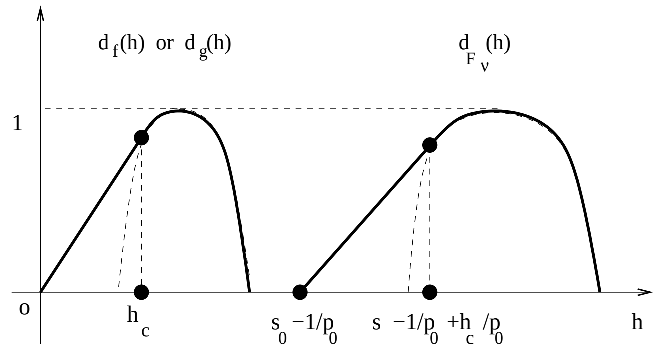

FIGURE 1. Multifractal spectra of the function series f and g (Left), and of the wavelet series F_ν (Right).

(2) If $h \geq h_c$ and **C3**(h) *holds, then $d_g(h) = d_\mu(h) = \tau_\mu^*(h)$.*

The behavior of g at the points x that have $\delta_x = +\infty$ cannot be precisely given.

An interesting fact is that, although the two function series we introduced do not fulfill any (known) multifractal formalism, their multifractal spectrum can be computed.

The proofs of Theorem 1.1 and Theorem 1.2 are different. This is due to the presence of the dilation parameter $b^{j\log(j)}$ in $\chi((x-kb^{-j})b^{j\log(j)})$. Indeed, the size of the support of the function $x \mapsto \chi((x-kb^{-j})b^{j\log j})$ converges much faster to 0 than the size of the support of $x \mapsto \phi(b^jx-k)$. This is where the notion of localized "pulse" comes from.

We finally propose in Section 5 a simple construction of multifractal functions based on wavelets. If the set of functions $\{\psi_{j,k}\}_{j,k\in\mathbf{Z}^2}$ forms a wavelet basis of $L^2(\mathbf{R})$ built as in [17], one defines for any positive real numbers s_0, p_0 such that $s_0 - \frac{1}{p_0} > 0$ the wavelet series

$$F_\nu(x) = \sum_{j\geq 1}\sum_{k\in\mathbf{Z}} d_{j,k}^\nu \psi_{j,k}(x), \tag{1.7}$$

where the wavelet coefficients $d_{j,k}^\nu$ are prescribed and equal to

$$d_{j,k}^\nu = 2^{-j(s_0-\frac{1}{p_0})}|\nu([k2^{-j},(k+1)2^{-j}))|^{\frac{1}{p_0}}. \tag{1.8}$$

Theorem 1.3. *Let μ be a positive Borel measure whose support is $[0,1]$, and assume that* **C1** *holds for μ. Let $q_c = \inf\{q\in\mathbf{R} : \tau_\mu(q) = 0\}$ and $h_c = \tau_\mu'(q_c^-)$.*

(1) If $h_c > 0$ and **C2**(h_c) *holds, for any $h \in [s_0 - \frac{1}{p_0}, s_0 - \frac{1}{p_0} + \frac{h_c}{p_0})$, one has $d_{F_\nu}(h) = q_c(h-(s_0-\frac{1}{p_0}))$.*

(2) If $h \geq h_c$ and **C3**(h) *holds, then $d_{F_\nu}(s_0 - \frac{1}{p_0} + \frac{h}{p_0}) = d_\mu(h) = \tau_\mu^*(h)$.*

Theorem 1.3 is a direct consequence of a work achieved in [2], and uses that ν satisfies a multifractal formalism for measures. The main difference with the two previous function series we proposed is that the function F_ν has a strictly positive global Hölder regularity ($F_\nu \in C^{s_0-\frac{1}{p_0}-\varepsilon}(\mathbf{R})$ for every $\varepsilon > 0$). Its multifractal spectrum starts at $(s_0 - \frac{1}{p_0}, 0)$ instead of $(0,0)$.

2. Definition and recalls

We first need to introduce some notations.

For $j \geq 1$ and $k \in [0, \dots, b^j - 1]$, one sets $I_{j,k} = [kb^{-j}, (k+1)b^{-j})$, $I^+_{j,k} = [(k+1)b^{-j}, (k+2)b^{-j}) = I_{j,k} + b^{-j}$ and $I^-_{j,k} = [(k-1)b^{-j}, kb^{-j}) = I_{j,k} - b^{-j}$.

If $x \in (0,1)$, $\forall j \geq 1$, $I_j(x)$ denotes the unique b-adic interval of length b^{-j}, semi-open to the right, that contains x, $I^+_j(x) = I_j(x) + b^{-j}$ and $I^-_j(x) = I_j(x) - b^{-j}$.

For each $j \geq 1$, $k_{j,x}$ is the unique integer such that $I_j(x) = [k_{j,x}b^{-j}, (k_{j,x}+1)b^{-j})$. One also denotes $k^+_{j,x} = k_{j,x} + 1$ and $k^-_{j,x} = k_{j,x} - 1$.

For the rest of the paper, the convention $\log(0) = -\infty$ is adopted.

Let us also recall that if $x \in \mathbf{R}$ and $\delta \geq 1$, x is said to be δ-approximated if there exists an infinite number of b-adic numbers kb^{-j} such that $|kb^{-j} - x| \leq b^{-j\delta}$. With each x can be associated its approximation rate δ_x defined by

$$\delta_x = \sup\{\delta : x \text{ is } \delta\text{-approximated}\}. \tag{2.1}$$

One always has $\delta_x \geq 1$, and it is shown in [8, 12] for example that the set $\{x \in \mathbf{R} : \delta_x = \delta\}$ has a Hausdorff dimension equal to $\frac{1}{\delta}$.

2.1. Local regularity of measures and functions.

In this work the local regularity of a measure μ is studied as follows.

Definition 2.1. Let μ be a positive Borel measure, and $x_0 \in [0,1]$. One sets

$$h_\mu(x_0) = \liminf_{r \to 0+} \frac{\log \mu(B(x_0,r))}{\log |B(x_0,r)|} = \liminf_{j \to +\infty} \frac{\log \mu(B(x_0,b^{-j}))}{\log |B(x_0,b^{-j})|}. \tag{2.2}$$

$|B|$ denotes the diameter of the set B.

Definition 2.2. For every measure μ and for every $\alpha \geq 0$, $E^\mu_\alpha = \{x : h_\mu(x) = \alpha\}$. The mapping $d_\mu : \ \alpha \geq 0 \mapsto \dim(E^\mu_\alpha)$ is the multifractal spectrum of μ.

$\dim E$ denotes the Hausdorff dimension of the set E.

The Legendre spectrum is useful in multifractal analysis, since it is more tractable than the multifractal spectrum d_μ. It also yields upper bounds of d_μ, as shown by the following proposition.

Proposition 2.3. *Let μ be a positive Borel measure, and let $\alpha \geq 0$.*

(1) One has $d_\mu(\alpha) \leq \tau^*_\mu(\alpha)$. If $\tau^*_\mu(\alpha) < 0$ then $E^\mu_\alpha = \emptyset$.
(2) If $\alpha \in [0, \tau^*_\mu(0^+)]$ then $\dim \bigcup_{\alpha' \leq \alpha} E^\mu_{\alpha'} \leq \tau^*_\mu(\alpha)$.
(3) If $\alpha \geq \tau^*_\mu(0^+)$ then $\dim \bigcup_{\alpha' \geq \alpha} E^\mu_{\alpha'} \leq \tau^*_\mu(\alpha)$.

This is deduced from Theorem 1 of [5], Proposition 2.5 of [18], Theorem 1 of [15] and Lemma 4.2 of [1].

Let us now focus on the function's frame. The local regularity of a function f at a point x_0 is often measured by the pointwise Hölder exponent h_f.

Definition 2.4. Let I be a non-trivial interval of $\mathbf{R}$, x_0 an interior point of I, and h a positive real number with $h \notin \mathbf{N}$. A function $f : I \to \mathbf{R}$ belongs to $C^h_{x_0}$ if and only if there exist a constant C and a polynomial P of degree smaller than $[h]$ such that for all x close enough to x_0

$$|f(x) - P(x - x_0)| \leq C|x - x_0|^h.$$

The *pointwise Hölder exponent* of f at x_0 is defined by $h_f(x_0) = \sup\{h : f \in C^h_{x_0}\}$.

Definition 2.5. Let I be a non trivial interval of $\mathbf{R}$ and $f : I \to \mathbf{R}$. Let us introduce $\forall\, h \geq 0$ the level sets

$$E_h^f = \{x \in \mathrm{Int}(I) : \ h_f(x) = h\}.$$

The mapping $d_f : h \geq 0 \mapsto \dim(E_h^f)$ is called the multifractal spectrum of f.

2.2. **Recalls on the multifractal spectrum of the measure ν.** If μ is a positive Borel measure with support equal to $[0,1]$, the associated measure ν is

$$\nu = \sum_{j\geq 1} \frac{1}{j^2} \sum_{k=0}^{b^j-1} \mu([kb^{-j},(k+1)b^{-j}))\, \delta_{kb^{-j}}. \tag{2.3}$$

The two following propositions, proved in [3], give a clue on the relation between the Hölder exponent of ν at a point x, the Hölder exponent of the measure μ at x and the approximation rate of x by the b-adic numbers.

Proposition 2.6. *Let $x \in E_\alpha^\mu$ for some $\alpha \geq 0$, and assume that δ_x defined by (2.1) is finite. Then*

$$\frac{\alpha}{\delta_x} \leq h_\nu(x) \leq \alpha.$$

Conditions **C1-3**, the definitions of which can be found in [3], will not be detailed here. Let us however precise that they are related to an ubiquity property, which is summarized in item (3) of next Proposition 2.8. This property allows to prove the non-emptiness and to lower bound the Hausdorff dimension of sets of points that are well-approximated by b-adic numbers kb^{-j} such that $\mu([kb^{-j},(k+1)b^{-j}))$ is controlled. As mentioned in the introduction, these conditions are satisfied by all the classical classes of multifractal measures (see [4]).

Definition 2.7. Let $h > 0$, and $x \in E_h^\nu$. Assume **C1** holds for μ. Then $\forall \varepsilon > 0$, there exist $\alpha \geq 0$ and $\delta \geq 1$ such that $\frac{\alpha}{\delta} \leq h + \varepsilon$ and $\eta < \varepsilon$, and an infinite number of irreducible b-adic numbers kb^{-j} that verify

$$b^{-j(\alpha+\eta)} \leq \mu([kb^{-j},(k+1)b^{-j})) \leq b^{-j(\alpha-\eta)} \text{ and } |kb^{-j} - x| \leq b.b^{-j\delta}. \tag{2.4}$$

Let us define for $\alpha > 0$, $\delta \geq 1$

$$E_{\alpha,\delta}^\mu = \left\{ x : \begin{array}{c} \text{There exist an infinite sequence of } b\text{-adic numbers } k_n b^{-j_n} \text{ that verify} \\ |k_n b^{-j_n} - x| \leq b^{-j_n\delta} \text{ and } \liminf_{n\to+\infty} \frac{\log \mu(I_{j_n,k_n})}{\log |I_{j_n,k_n}|} = \alpha \end{array} \right\}. \tag{2.5}$$

Next proposition recalls results from [3] (τ_μ and its Legendre transform τ_μ^* are defined in (1.2) and (1.3)).

Proposition 2.8. *Let μ be a positive Borel measure whose support is $[0,1]$, and assume that* **C1** *holds for μ. Let $q_c = \inf\{q \in \mathbf{R} : \tau_\mu(q) = 0\}$ and $h_c = \tau_\mu'(q_c^-)$.*

(1) For every $\alpha > 0$ and every $\delta \geq 1$, $\dim E_{\alpha,\delta}^\mu \leq \frac{\tau_\mu^*(\alpha)}{\delta}$.

(2) Let $F : \mathbf{R} \to \mathbf{R}$ be a positive strictly increasing continuous function, such that $\lim_{+\infty} F(x) = +\infty$. Let $G_h(F)$ be the set of points defined by

$$G_h(F) = \left\{ x : \begin{array}{c} \forall \varepsilon > 0,\ \exists\, \alpha \geq 0,\ \exists\, \delta \geq 1 \text{ such that } \frac{F(\alpha)}{\delta} \leq h + \varepsilon \text{ and} \\ (2.4) \text{ holds for an infinite number of irreducible } b\text{-adic numbers} \end{array} \right\}.$$

Then

$$\dim G_h(F) \leq h \sup_{\alpha : F(\alpha) \geq h} \frac{\sup_{\alpha' \leq \alpha} \tau_\mu^*(\alpha')}{F(\alpha)}. \tag{2.6}$$

(3) If **C2**(h) *holds, then for every* $\delta \geq 1$ *there exist a positive Borel measure* m_δ *and a subset* S *of* $E^\nu_{\frac{h}{\delta}}$ *such that* $m_\delta(S) > 0$*, and such that for every set* $E \subset [0,1]$ *such that* $\dim E < \frac{\tau^*_\mu(h)}{\delta}$, $m_\delta(E) = 0$.

In particular, $\dim E^\nu_{\frac{h}{\delta}} \geq \dim S \geq \frac{\tau^*_\mu(h)}{\delta}$.

(4) If $h \geq h_c$ *and* **C3**(h) *holds, then there exist a positive Borel measure* m_h *and a subset* S *of* $E^\mu_h \cap \{x : \delta_x = 1\} \cap E^\nu_h$ *such that* $m_h(S) > 0$*, and such that for every set* $E \subset [0,1]$ *with* $\dim E < \tau^*_\mu(h)$, $m_h(E) = 0$.

In particular, $\dim(E^\mu_h \cap \{x : \delta_x = 1\} \cap E^\nu_h) \geq \dim S \geq \tau^*_\mu(h)$.

The next theorem, proved in [3], allows to compute the multifractal spectrum of the measure ν (2.3). We emphasize one more time that it relies on the measure-conditioned ubiquity property of the initial measure μ.

Theorem 2.9. *Let* μ *be a positive Borel measure whose support is* $[0,1]$*, and assume that* **C1** *holds for* μ*. Let* ν *be the measure given by formula (2.3). Let* $q_c = \inf\{q \in \mathbf{R} : \tau_\mu(q) = 0\}$*, and* $h_c = \tau'_\mu(q_c^-)$.

(1) If $h_c > 0$*, for every* $h \in [0, h_c]$*, one has* $d_\nu(h) \leq q_c h$*. If moreover* **C2**(h_c) *holds, then for every* $h \in [0, h_c]$ $d_\nu(h) = q_c h$*, and the multifractal formalism holds at* h.

(2) If $h \geq h_c$*,* $d_\nu(h) \leq \tau^*_\mu(h)$*. If moreover* **C3**$(h)$ *holds, then* $d_\mu(h) = d_\nu(h) = \tau^*_\mu(h) = \tau^*_\nu(h)$*, and the multifractal formalism holds at* h.

3. Function series with multifractal jumps

Let us recall (1.4)

$$f(x) = \sum_{j=1}^{+\infty} f_j(x) \text{ where } f_j(x) = \frac{1}{j^2} \sum_{k=0}^{b^j-1} \mu(I^+_{j,k})\phi(b^j x - k)$$

with $\phi(x) = x$ if $x \in [0,1)$, and $\phi(x) = 0$ elsewhere. Remark that for each scale j, there exists only one integer k (equal to $k_{j,x}$) such that $\phi(b^j x - k) \neq 0$. Hence for every x, $f(x) = \sum_{j=1}^{+\infty} \frac{1}{j^2}\mu(I_j(x)^+)\phi(b^j x - k)$.

Remark that for a fixed b-adic number Kb^{-J}, the sizes of the jumps of the function f_j's at Kb^{-J} (which is equal to $\frac{1}{j^2}\mu([Kb^{-J}, Kb^{-J} + b^{-j}))$) form a positive sequence decreasing when $j \to +\infty$. Hence by construction, the size of the jump of f at Kb^{-J} is always greater than $\frac{1}{J^2}\mu(I_{J,K})$.

3.1. **First properties of the function series.**

The following lemma, found in [9], will be of great use.

Lemma 3.1. *Let us assume that a function* f *is discontinuous on a dense set of* $\mathbf{R}$*. For a fixed* $x \in \mathbf{R}$*, let us assume that there exists a sequence* $\{r_n\}_n$ *converging to* x *such that for every* n*,* f *has right and left limits* $f(r_n^+)$ *and* $f(r_n^-)$ *at* r_n*, and* $|f(r_n^+) - f(r_n^-)| = s_n > 0$*. Then*

$$h_f(x) \leq \liminf_{n \to +\infty} \frac{|\log s_n|}{|\log |r_n - x_0||}.$$

As a consequence,

Proposition 3.2. *Let* f *and* ν *be given by (1.4) and (2.3). For every* $x \in [0,1] \cap \operatorname{supp}(\mu)$*,* $h_f(x) \leq h_\nu(x)$.

Proof. If x is a b-adic number, then ν and f clearly have the same regularity at x, i.e. 0.

Let us thus assume that $x \in [0,1]$ is not a b-adic number, and denote $h_\nu(x) = h$.

Assume first that $\delta_x < +\infty$. Let us compute the pointwise Hölder exponent $h_f(x)$ of f at x.

In order to upper bound $h_f(x)$, we combine Proposition 2.7 and Lemma 3.1. Indeed, let $\varepsilon > 0$. By Proposition 2.7, one can find $\alpha, \delta \geq 1$ such that $\frac{\alpha}{\delta} \leq h + \varepsilon$, and an infinite number of irreducible b-adic numbers kb^{-j} such that (2.4) holds.

Let us denote $r_n = k_n b^{-j_n}$ such a sequence of b-adic numbers, with $j_n \to +\infty$. As a consequence, $k_n b^{-j_n} \to x$ when $n \to +\infty$. Note first that since r_n is irreducible, one has for every $j < j_n$ $|f_j(r_n^+) - f_j(r_n^-)| = 0$. Second, by the remark above, for every n, $s_n = |f(r_n^+) - f(r_n^-)| \leq \frac{1}{j_n^2}\mu(I_{j_n,k_n})$.

One applies Lemma 3.1 to get that $h_f(x) \leq \liminf_{n\to+\infty} \frac{|\log s_n|}{|\log|r_n - x||}$. But $|\log s_n| \leq |\log \mu([k_n b^{-j_n}, (k_n+1)b^{-j_n}))| + 2\log j_n$ which by (2.4) is bounded by $(\alpha+\varepsilon) j_n \log b + 2\log j_n$. Moreover, still by (2.4), $|\log|r_n - x|| \geq |\log b b^{-j_n\delta}|$. Thus

$$h_f(x) \leq \liminf_{n\to+\infty} \frac{(\alpha+\varepsilon)j_n \log b + 2\log j_n}{|\log b b^{-j_n\delta}|} \leq h + \varepsilon + \frac{\varepsilon}{\delta}.$$

Using (2.4) and letting $\varepsilon \to 0$ yields the upper bound $h_f(x) \leq h = h_\nu(x)$.

The case $\delta_x = +\infty$ is obvious. □

Let us now move to the lower bound of the pointwise Hölder exponent of f at x.

Proposition 3.3. *Let f be given by (1.4), and let ν be given by (2.3). For every $x \in (0,1)$ with an approximation rate $\delta_x < \infty$,*

- if $h_\nu(x) \leq 1$, $h_f(x) \geq h_\nu(x)$.

- if $h_\nu(x) > 1$, $h_f(x) \geq 1 + \frac{h_\mu(x)-1}{\delta_x}$.

Proof. By definition of δ_x, there exists a scale J_1 such that $j \geq J_1$ implies

$$\forall k,\ \ |kb^{-j} - x| \geq b^{-j(\delta_x+\varepsilon)}. \tag{3.1}$$

Moreover, by definition of $h = h_\nu(x)$ and $h_\mu(x)$, there exists a scale J_2 such that $j \geq J_2$ implies

$$\mu(B(x, b^{-j})) \leq b^{-j(h_\mu(x)-\varepsilon)}. \tag{3.2}$$

and

$$\nu(B(x, b^{-j})) \leq b^{-j(h-\varepsilon)}. \tag{3.3}$$

We denote by J_x the integer $\max(2\delta_x J_1, J_2)$. Let $r > 0$, and let J_r be the unique integer such that

$$b^{-J_r} \leq r < b^{-(J_r-1)}. \tag{3.4}$$

We assume that r is close enough to 0 so that $J_r \geq J_x$. We denote by J the smallest scale such that there exists an integer K with $Kb^{-J} \in [x - r, x]$. One has by (3.1) $\frac{J_r}{\delta_x+\varepsilon} \leq J \leq J_r$.

• **First case:** $h \leq 1$**.** We want to upper bound $|f(x) - f(x-r)| = |\sum_{j=1}^{+\infty} \frac{1}{j^2} f^{(j)}(x)|$, where

$$f^{(j)}(x) = \sum_{k=0}^{b^j-1} \mu(I_{j,k}^+)(\phi(b^j x - k) - \phi(b^j(x-r) - k)).$$

- When $j < J$, x and $x-r$ both belong to $I_j(x) = [k_{j,x}b^{-j}, (k_{j,x}+1)b^{-j})$, and

$$f^{(j)}(x) = \mu(I_j(x)^+)(\phi(b^j x - k) - \phi(b^j(x-r) - k)),$$

which equals $\mu(I_j(x)^+)b^j r$ since ϕ is a linear function with slope 1. Hence

$$(A) = \left|\sum_{j=1}^{J-1} \frac{1}{j^2} f^{(j)}(x)\right| \leq \sum_{j=1}^{J-1} \frac{1}{j^2}\mu(I_j(x)^+)b^j r.$$

Using (3.2), one sees that there exists a constant C so that $\forall j$, $\mu(I_j(x)^+) \leq Cb^{-j(h_\mu(x)-\varepsilon)}$. Hence, remembering that $h \leq h_\mu(x)$ and $1-(h-\varepsilon) > 0$, one has

$$\begin{aligned}(A) &\leq \sum_{j=1}^{J} \frac{1}{j^2}\mu(I_j(x)^+)b^j r \leq Cr\sum_{j=1}^{J} b^j b^{-j(h_\mu(x)-\varepsilon)} \leq Cr\sum_{j=1}^{J} b^{j(1-(h-\varepsilon))}\\ &\leq Crb^{J(1-(h-\varepsilon))} \leq Crb^{J_r(1-(h-\varepsilon))} \leq Crr^{h-\varepsilon-1} \leq Cr^{h-\varepsilon},\end{aligned}$$

where (3.4) has been used to get $b^J \leq b^{J_r} \leq Cr$.

- When $J \leq j \leq J_r - 1$, $k_{j,x}b^{-j} = Kb^{-J}$ is the unique b-adic number of generation j in the interval $[x-r, x]$. Moreover, $x \in I_j(x)$ and $x - r \in I_j(x)^-$, which implies $|x - k_{j,x}b^{-j}| \leq r$. As a consequence,

$$\begin{aligned}|f^{(j)}(x)| &\leq \mu(I_j(x))\phi(b^j(x-r) - (k_{j,x}-1)) + \mu(I_j(x)^+)\phi(b^j x - k_{j,x})\\ &\leq \mu(I_j(x)) + b^j r\mu(I_j(x)^+),\end{aligned}$$

and thus the sum (B)= $\left|\sum_{j=J}^{J_r-1} \frac{1}{j^2} f^{(j)}(x)\right|$ is bounded by

$$(B) \leq \sum_{j=J}^{J_r-1} \frac{1}{j^2}\left(\mu(I_j(x)) + \mu(I_j(x)^+)b^j r\right) = \sum_{j=J}^{J_r-1} \frac{1}{j^2}\mu(I_j(x)) + r\sum_{j=J}^{J_r-1} \frac{b^j}{j^2}\mu(I_j(x)^+).$$

By definition of the measure ν, $\sum_{j=J}^{J_r-1} \frac{1}{j^2}\mu(I_j(x)) \leq \nu(B(x,r))$, which is lower than $Cr^{h-\varepsilon}$ by (3.3). On the other hand, using (3.4),

$$\begin{aligned}r\sum_{j=J}^{J_r-1} \frac{1}{j^2}\mu(I_j(x)^+)b^j &\leq Cb^{-(J_r-1)}\sum_{j=J}^{J_r-1} b^{-j(h_\mu(x)-\varepsilon)}b^j \leq Cb^{-(J_r-1)}\sum_{j=J}^{J_r-1} b^{-j(h-\varepsilon)}b^j\\ &\leq Cb^{-(J_r-1)}b^{J_r(1-(h-\varepsilon))} \leq Cb^{-J_r(h-\varepsilon)}.\end{aligned}$$

Hence $(B) \leq Cr^{h-\varepsilon}$.

- Finally, when $j \geq J_r$, x and $x-r$ do not belong to the same b-adic interval. Hence $\left|f^{(j)}(x)\right| \leq \mu(I_j(x)^+) + \mu(I_j(x+r)^+) \leq 2\mu(B(x,br))$, and

$$(C) = \left|\sum_{j\geq J_r} \frac{1}{j^2} f^{(j)}(x)\right| \leq \sum_{j\geq J_r} \frac{1}{j^2} 2\mu(B(x,br)) \leq C\mu(B(x,br)) \leq Cr^{h_\mu(x)-\varepsilon}.$$

The discussion is the same for $|f(x+r) - f(x)|$.

Grouping the results gives $|f(x) - f(x-r)| \leq (A) + (B) + (C) \leq Cr^{h-\varepsilon}$, and this implies that $h_f(x) \geq h - \varepsilon$, for every $\varepsilon > 0$. Hence the lower bound.

• **Second case:** $h > 1$**.** Let us define the polynomial function of degree 1 defined by

$$P : y \mapsto \sum_{j\geq 1} \frac{1}{j^2}\mu(I_j(x)^+)b^j y.$$

We shall use this polynomial to obtain the Hölder exponent of f at x (remember (2.4)).

Let $r>0$. Let us assume that ε is small enough so that $h-\varepsilon>1$, and let us now upper bound $f(x)-f(x-r)-P(r)=\sum_{j=1}^{+\infty}\frac{1}{j^2}g^{(j)}(x)$, where

$$g^{(j)}(x)=\sum_{k=0}^{b^j-1}\mu(I_{j,k}^+)\left(\phi(b^jx-k)-\phi(b^j(x-r)-k)\right)-\mu(I_j(x)^+)b^jr.$$

- When $j<J$, $x-r$ and x belong to the same b-adic interval $I_j(x)$, and

$$\begin{aligned} g^{(j)}(x) &= \mu(I_j(x)^+)(\phi(b^jx-k_{j,x})-\phi(b^j(x-r)-k_{j,x}))-\mu(I_j(x)^+)b^jr \\ &= \mu(I_j(x)^+)((b^jx-k_{j,x})-(b^j(x-r)-k_{j,x})-b^jr)=0. \end{aligned}$$

- When $J\le j\le J_r-1$, $x\in I_j(x)$ and $x-r\in I_j(x)^-$. Hence

$$\begin{aligned} g^{(j)}(x) &= \mu(I_j(x)^+)b^j(x-k_{j,x}b^{-j})-\mu(I_j(x))b^j((x-r))-k_{j,x}b^{-j})-\mu(I_{j,k}^+)b^jr \\ &= (\mu(I_j(x)^+)-\mu(I_j(x)))b^j((x-r)-k_{j,x}b^{-j}). \end{aligned}$$

Since $x\in E^\mu_{h_\mu(x)}$, $|\mu(I_j(x)^+)-\mu(I_j(x))|\le Cb^{-j(h_\mu(x)-\varepsilon)}$. Moreover, $|(x-r)-k_{j,x}b^{-j}|\le r$.

By Proposition 2.6, $h_\mu(x)-\varepsilon\ge h-\varepsilon>1$. Combining this with (3.4) and $\frac{J_r}{\delta_x+\varepsilon}\le J\le J_r$, one gets

$$\begin{aligned} \sum_{j=J}^{J_r-1}\frac{1}{j^2}|g^{(j)}(x)| &\le \sum_{j=J}^{J_r-1}Cb^{-j(h_\mu(x)-\varepsilon)}b^jr\le Cr\sum_{j=J}^{J_r-1}b^{j(1-(h_\mu(x)-\varepsilon))} \\ &\le Cb^{-J_r}b^{J(1-(h_\mu(x)-\varepsilon))} \\ &\le Cb^{-J_r(1+\frac{h_\mu(x)-\varepsilon-1}{\delta_x+\varepsilon})}\le Cr^{1+\frac{h_\mu(x)-\varepsilon-1}{\delta_x+\varepsilon}}. \end{aligned} \tag{3.5}$$

(the constant C depends on x and $h_\mu(x)$, but not on J or r.)

- Finally, when $j\ge J_r$, one has

$$|g^{(j)}(x)|\le b^jr\mu(I_j(x)^+)+\mu(I_j(x)^+)+\mu(I_j^+(x-r))\le\mu(I_j(x)^+)b^jr+2\mu(B(x,br)).$$

Hence, using again (3.4), one obtains

$$\begin{aligned} \sum_{j\ge J_r}\frac{1}{j^2}|g^{(j)}(x)| &\le \sum_{j=J_r}^{+\infty}\frac{1}{j^2}b^jr\mu(I_j(x)^+)+2\frac{1}{j^2}\mu(B(x,br)) \\ &\le C\sum_{j=J_r}^{+\infty}b^jrb^{-j(h_\mu(x)-\varepsilon)}+C\sum_{j=J_r}^{+\infty}\frac{1}{j^2}\mu(B(x,br)) \\ &\le Cr^{h_\mu(x)-\varepsilon}. \end{aligned} \tag{3.6}$$

Grouping all the results yields

$$|f(x)-f(x-r)-P(r)|\le Cr^{1-\frac{h_\mu(x)-\varepsilon-1}{\delta_x+\varepsilon}}+Cr^{h_\mu(x)-\varepsilon}\le Cr^{1-\frac{h_\mu(x)-\varepsilon-1}{\delta_x+\varepsilon}}.$$

The proof is the same for $r<0$, and thus $h_f(x)\ge 1+\frac{h_\mu(x)-\varepsilon-1}{\delta_x+\varepsilon}$. Letting ε go to zero yields $h_f(x)\ge 1+\frac{h_\mu(x)-1}{\delta_x}$. □

Remark that if $\delta_x=1$, Propositions 2.6, 3.2 and 3.3 together give $h_f(x)=h_\nu(x)=h_\mu(x)$.

We now state a proposition which can be deduced from the one of Lemma 5 and Proposition 5 of [3] and from the estimates obtained in (3.5) and (3.6).

Proposition 3.4. *Let F denote the function $x \mapsto x-1$. Assume that* **C1** *holds for μ. If $h>1$ then $E_h^f \subset G_{h-1}(F)$. Moreover, if $x \in G_{h-1}(F)$ then $h_f(x) \leq h$.*

3.2. **Computation of the multifractal spectrum: Proof of Theorem 1.1.** Let $h>0$. We distinguish three cases

• **If** $h \leq 1$**:** Propositions 3.2 and 3.3 imply that for all $0 < h' \leq 1$, $\forall x \in E_{h'}^\nu$, one has $h' = h_\nu(x) = h_f(x)$. Hence $E_h^\nu \subset E_h^f$ and if $x \in E_h^f$ verifies $h_\nu(x) > h$, one has in fact $h_\nu(x) > 1$ and $h \geq 1 + \frac{h_\mu(x)-1}{\delta_x} > 1$ by Proposition 3.3. Consequently $E_h^\nu = E_h^f$ and $d_f(h) = d_\nu(h)$. Theorem 2.9 gives the result.

• **If** $\tau_\mu'(0^+) > 1$ **and** $1 < h < \tau_\mu'(0^+)$**:** By Proposition 3.4, $E_h^f \subset G_{h-1}(F)$, where F is the function $F(x) = x-1$. Applying (2.6) yields

$$\dim G_{h-1}(F) \leq (h-1) \sup_{\alpha-1 \geq h-1} \frac{\sup_{\alpha' \leq \alpha} \tau_\mu^*(\alpha')}{\alpha - 1} = (h-1) \sup_{\alpha \geq h} \frac{\sup_{\alpha' \leq \alpha} \tau_\mu^*(\alpha')}{\alpha - 1}. \tag{3.7}$$

For $\alpha \geq \tau_\mu'(0^+)$, one has $\sup_{\alpha' \leq \alpha} \tau_\mu^*(\alpha') = \tau_\mu^*(\tau_\mu'(0^+)) (=1)$. Moreover, $\alpha \mapsto \frac{1}{\alpha-1}$ is decreasing on $(1,\infty)$. Since $h \leq \tau_\mu'(0^+)$, the supremum in (3.7) is always reached for some exponent $\alpha \leq \tau_\mu'(0^+)$, and

$$\dim G_{h-1}(F) \leq (h-1) \sup_{h \leq \alpha \leq \tau_\mu'(0^+)} \frac{\sup_{\alpha' \leq \alpha} \tau_\mu^*(\alpha')}{\alpha - 1}.$$

Eventually one remarks that $\sup_{\alpha' \leq \alpha} \tau_\mu^*(\alpha') = \tau_\mu^*(\alpha)$ when $\alpha \leq \tau_\mu'(0^+)$ (this is the increasing part of the spectrum). Thus $\dim G_{h-1}(F) \leq \sup_{h \leq \alpha \leq \tau_\mu'(0^+)} (h-1) \frac{\tau_\mu^*(\alpha)}{\alpha-1}$. It is easy to see that for every $\alpha \geq h$, $\frac{h-1}{\alpha-1} \leq \frac{h}{\alpha}$. Hence $\dim G_{h-1}(F) \leq h \sup_{h \leq \alpha \leq \tau_\mu'(0^+)} \frac{\tau_\mu^*(\alpha)}{\alpha} = h \sup_{h \leq \alpha} \frac{\tau_\mu^*(\alpha)}{\alpha}$.

The following manipulation is done in [3]. If $h \leq h_c$, $\sup_{h \leq \alpha} \frac{\tau_\mu^*(\alpha)}{\alpha} = \frac{\tau_\mu^*(h_c)}{h_c} = q_c$, and $\dim G_{h-1}(F) \leq q_c h = d_\nu(h)$. If $h \geq h_c$, $\sup_{h \leq \alpha} \frac{\tau_\mu^*(\alpha)}{\alpha} = \frac{\tau_\mu^*(h)}{h}$, and $\dim G_{h-1}(F) \leq h \frac{\tau_\mu^*(h)}{h} = \tau_\mu^*(h) = d_\nu(h)$. In both cases, $\dim E_h^f \leq d_\nu(h)$.

Let us move to the lower bound.

Assume first that $h_c > 1$ and $h \leq h_c$. Let $\delta = \frac{h_c}{h}$. If **C2(**h_c**)** holds, let us consider the measure m_δ and the set $S \subset E_{\frac{h_c}{\delta}}^\nu$ invoked in item (3) of Proposition 2.8.

Let $x \in S \setminus \bigcup_{1<h'<h} G_{h'-1}(F)$. Since $h_\nu(x) = h > 1$, it follows from Propositions 3.2, 3.3 and 3.4 that $x \in E_h^f$. Moreover, we deduce from our study of the upper bound of $\dim E_h^f$ that for every $h' \in (1,h)$, one has $\dim G_{h'-1}(F) \leq d_\nu(h) = q_c h$. Using the properties of the measure m_δ, one obtains

$$m_\delta \left(\dim \bigcup_{1<h'<h} G_{h'-1}(F) \right) = 0.$$

Consequently, by definition of S, the subset $S \setminus \bigcup_{1<h'<h} G_{h'-1}(F)$ of E_h^f is of positive m_δ-measure, and $\dim E_h^f \geq \dim E_h^\nu$, which yields the result.

Let us assume now that $h \in [h_c, \tau_\mu'(0^+)]$, and that **C3(**$h$**)** holds. We denote by m_h and $S \subset E_h^\nu$ the measure and the set invoked in item (4) of Proposition 2.8. Since for every $x \in S$, $\delta_x = 1$, then $h_f(x) = h_\nu(x)$. Thus $S \subset E_h^f$, and $m_h(E_h^f) \geq m_h(S) > 0$ which yields $\dim E_h^f \geq \dim E_h^\nu$.

• **If** $h \geq \tau_\mu'(0^+)$**:** In this decreasing part of the spectrum there is no problem with the upper bound. Indeed, Proposition 3.2 yields $E_h^f \subset \bigcup_{\alpha\geq h} E_\alpha^\nu$, and by Proposition 2.8 $\dim \bigcup_{\alpha\geq h} E_\alpha^\nu \leq \tau_\mu^*(h) = \tau_\mu^*(h) = d_\nu(h)$. Hence, $d_f(h) \leq d_\nu(h)$.

In order to find the lower bound, the same arguments as just above (when $h \in [h_c, \tau_\mu'(0^+)]$) apply here: Item (4) of Proposition 2.8 yields a set $S \subset E_h^f$ such that $m_h(S) > 0$. Hence $\dim E_h^f \geq \dim S = d_\nu(h)$. Theorem 1.1 follows.

3.3. **Deforming the spectra.** We follow the same ideas as in [3]. Introducing two parameters γ and σ allows to build measures with a spectrum that can be changed according to the values of γ and σ.

We give the results for the corresponding function series f without proof. Indeed they are easy adaptations of the work achieved for f in Section 3 and of the following theorem proved in [3].

Theorem 2.9.' *Let μ be a positive Borel measure whose support is $[0,1]$, and assume that* **C1** *holds for μ. Let $\gamma \geq 0$ and $\sigma \geq 1$. Let $q_{\gamma,\sigma} = \inf\{q \in \mathbf{R} : \tau_\mu(\sigma q) + \gamma q = 0\}$, and $h_{\gamma,\sigma} = \sigma\tau_\mu'(\sigma q_{\gamma,\sigma}^-) + \gamma$. One has $q_{\gamma,\sigma} \in (0,1]$ and $0 \leq h_{\gamma,\sigma} \leq q_{\gamma,\sigma}^{-1}$. Let*

$$\nu_{\gamma,\sigma} = \sum_{j=1}^{+\infty} \frac{b^{-j\gamma}}{j^2} \sum_{k=0}^{b^j-1} \mu(I_{j,k}^+)^\sigma \delta_{kb^{-j}}.$$

(1) If $h_{\gamma,\sigma} > 0$ and **(C2(**$\frac{h_{\gamma,\sigma}-\gamma}{\sigma}$**))** *holds, for any $h \in [0, h_{\gamma,\sigma}]$, $d_{\nu_{\gamma,\sigma}}(h) = q_{\gamma,\sigma} h$. Moreover, the multifractal formalism holds at h.*

(2) If $h \geq h_{\gamma,\sigma}$ and **C3(**$\frac{h-\gamma}{\sigma}$**)** *holds, then $d_{\nu_{\gamma,\sigma}}(h) = \tau_\mu^*\left(\frac{h-\gamma}{\sigma}\right)$ and the multifractal formalism holds at h.*

Let $f_{\gamma,\sigma}$ be the function series

$$f_{\gamma,\sigma}(x) = \sum_{j=1}^{+\infty} \frac{b^{-j\gamma}}{j^2} \sum_{k=0}^{b^j-1} \mu(I_{j,k}^+)^\sigma \phi(b^j x - k)$$

Theorem 1.1.' *Under the assumptions of Theorem 2.9.', one has*

(1) If $h_{\gamma,\sigma} > 0$ and **C2(**$\frac{h_{\gamma,\sigma}-\gamma}{\sigma}$**)** *holds, for any $h \in [0, h_{\gamma,\sigma}]$, $d_{f_{\gamma,\sigma}}(h) = q_{\gamma,\sigma} h$.*

(2) If $h \geq h_{\gamma,\sigma}$ and **C3(**$\frac{h-\gamma}{\sigma}$**)** *holds, then $d_{f_{\gamma,\sigma}}(h) = \tau_\mu^*\left(\frac{h-\gamma}{\sigma}\right)$.*

Theorem 1.1.' gives the possibility to play with the parameters σ and γ in order to obtain a large variety of multifractal spectra.

4. Infinite sums of pulses

Now consider the function g defined by (1.5). Pay attention to the fact that the sum is taken over the irreducible b-adic numbers.

We denote by $\chi_{j,k}(x)$ the function $x \mapsto \chi\big((x - kb^{-j})b^{j\log j}\big)$. The function $\chi_{j,k}$ is thus a dilated translated version of χ with a support included in $[kb^{-j} - b^{-j\log j}, kb^{-j} + b^{-j\log j}]$.

4.1. **Proof of Theorem 1.2.** Theorem 1.2 follows from the next proposition

Proposition 4.1. *Let $x \in [0,1]$ be such that $\delta_x < +\infty$. Assume that $\widetilde{\mathbf{C1}}$ holds for μ. Then $h_g(x) = h_\nu(x)$.*

Proof. We set $h = h_\nu(x)$. We first prove that $h_g(x) \geq h$. Let $\varepsilon > 0$. As in the proof of Proposition 3.3, there exists a scale J_1 such that for $j \geq J_1$, (3.1) and (3.3) together hold. Moreover, J_1 can also be chosen large enough so that $\log J_1 > \delta_x + 1$. This implies that

$$\sum_{j \geq J_1+1} \frac{1}{j^2} \sum_k \mu(I_{j,k}) \chi_{j,k}(x) = 0. \tag{4.1}$$

We set $J_x = J_1$ and $J'_x = 1 + [(J_x+1)(\delta_x + \varepsilon)]$.

Let $y \in \mathbf{R}$ such that $|x-y| \leq b^{-J'_x}$. There exists an integer $J_y \geq J'_x$ such that

$$b^{-J_y - 1} \leq |y - x| < b^{-J_y}. \tag{4.2}$$

Suppose that there exists $j \geq J_x$ and $0 \leq k < b^{-j}$ such that $|y - kb^{-j}| \leq b^{-j \log j}$, i.e. $\chi_{j,k}(y) \neq 0$. Then $|x - kb^{-j}| \leq |x-y| + |y - kb^{-j}| \leq b^{-J_y} + b^{-j\log j}$. One knows that $|x - kb^{-j}| \geq b^{-j(\delta_x + \varepsilon)}$. Hence $b^{-J_y} \geq b^{-j(\delta_x+\varepsilon)} - b^{-j \log j}$, which becomes greater than $\frac{1}{2} b^{-j(\delta_x+\varepsilon)}$ when J_x is large enough. This implies that $J_y \leq j(\delta_x + \varepsilon) + 1$, or equivalently that $j \geq \frac{J_y - 1}{\delta_x + \varepsilon} \geq J_x + 1$ since $J_y \geq J'_x$.

A consequence is that for every $j \in [J_x, \frac{J_y-1}{\delta_x+\varepsilon}]$, $\chi_{j,k}(y) = 0$ for every $k \in \mathbf{Z}$.

We set $J'_y = [\frac{J_y - 1}{\delta_x + \varepsilon}]$. Remark that there exists a constant C depending only on ε such that

$$J'_y \geq C \frac{J_y}{\delta_x}. \tag{4.3}$$

Due to the above remarks, one has $g(y) - g(x) = (A) + (B)$ where

$$\begin{aligned} (A) &= \sum_{j=1}^{J_x} \frac{1}{j^2} \sum_k \mu(I_{j,k}) (\chi_{j,k}(y) - \chi_{j,k}(x)) \\ (B) &= \sum_{j \geq J'_y + 1} \frac{1}{j^2} \sum_k \mu(I_{j,k}) \chi_{j,k}(y). \end{aligned}$$

- The first term (A) contains a finite number of non-zero terms. Moreover, when j and k are fixed, since $\chi \in C^{[D]+2}([-1,1])$, by definition of the pointwise Hölder exponent there exists a polynomial $P_{j,k}$ of degree equal or less than $[h]$ and two constants $\eta_{j,k}$ and $C_{j,k}$ such that

$$|x - z| \leq \eta_{j,k} \Rightarrow |\chi_{j,k}(z) - \chi_{j,k}(x) - P_{j,k}(z)| \leq C_{j,k} |x-z|^{[h]+1} \leq C_{j,k} |x - z|^h,$$

since $h \leq [D] + 1$. Hence, if P denotes the polynomial $\sum_{j=1}^{J_x} \frac{1}{j^2} \sum_k \mu(I_{j,k}) P_{j,k}$, if $\eta = \min_{j,k}(\eta_{j,k})$ and $C = \max_{j,k}(\frac{\mu(I_{j,k})}{j} C_{j,k})$, then

$$|x - y| \leq \eta \Rightarrow |(A) - P(y)| \leq C|x-y|^h.$$

- Consider now the term $(B) = \sum_{j \geq J'_y+1} \frac{1}{j^2} \sum_k \mu(I_{j,k}) \chi_{j,k}(y)$. At each scale j, only one term may be non-zero. Moreover, as noticed before, each time $\chi_{j,k}(y)$ is non-zero, one has $|kb^{-j} - x| \leq b^{-J'_y}$.

Let $j \geq J'_y$ be such that there exists k_y with $\chi_{j,k_y}(y) \neq 0$. One uses (4.3) and the definition of χ_{j,k_y} to get

$$|k_y b^{-j} - y| \leq b^{-j\log j} \leq b^{-J'_y \log J'_y} \leq b^{-J_y\left(C\frac{\log J_y - \log \frac{\delta_x}{C}}{\delta_x}\right)}.$$

We assume without loss of generality that J_x has been chosen large enough so that $C\frac{\log J_y - \log \frac{\delta_x}{C}}{\delta_x} \geq 1$. Using that $|x-y| \leq b^{-J_y}$, one gets

$$|k_y b^{-j} - x| \leq |k_y b^{-j} - y| + |y-x| \leq b^{-J_y} + b^{-J_y\left(C\frac{\log J_y - \log \frac{\delta_x}{C}}{\delta_x}\right)} \leq 2b^{-J_y}.$$

Hence $k_y b^{-j}$ belongs to the ball $B(x, 2b^{-J_y})$. Then, using (3.3), one gets that $\nu(B(x,2b^{-J_y})) \leq C(b^{-J_y})^{h-\varepsilon}$.

There exists a b-adic interval of length b^{-J_y}, namely L, such that $I_{j,k_y} \subset L \subset B(x, 2b^{-J_y})$. Remarking that one always has $\nu(L) \geq \frac{1}{(\log_c |L|)^2}\mu(L)$, this yields

$$\frac{1}{J_y^2}\mu(L) = \frac{1}{(\log_c |L|)^2}\mu(L) \leq \nu(L) \leq \nu(B(x, 2b^{-J_y})).$$

Consequently, $\mu(I_{j,k_y}) \leq C J_y^2 b^{-J_y(h-\varepsilon)}$. It follows that

$$\begin{aligned}(B) &\leq C\|\chi\|_\infty J_y^2 b^{-J_y(h-\varepsilon)} \sum_{j \geq J'_y} \frac{1}{j^2} = C\|\chi\|_\infty \frac{J_y^2}{J'_y} b^{-J_y(h-\varepsilon)} \\ &\leq C\|\chi\|_\infty \delta_x J_y b^{-J_y(h-\varepsilon)} \leq C_x |y-x|^{h-\varepsilon} |\log(|y-x|)|.\end{aligned}$$

Summing the terms (A) and (B) and substracting the polynomial P, one has as soon as $|x-y| \leq \min(\eta, b^{-J'_x})$

$$\begin{aligned}|g(x) - g(y) - P(y)| &\leq C|x-y|^{h+1} + 0 + Cx - y|^{h-\varepsilon} |\log(|x-y|)| \\ &\leq C|x-y|^{h-\varepsilon}|\log(|x-y|)|,\end{aligned}$$

for some constant C. This remains true for every $\varepsilon > 0$, hence $h_g(x) \geq h = h_\nu(x)$.

In order to prove the converse inequality (i.e. $h_g(x) \leq h_\nu(x)$), we use Proposition 2.7. Let $\varepsilon > 0$ as in the proof of Proposition 3.3. There exists a couple (α, δ) and an infinite number of b-adic numbers kb^{-j} with $j \geq J_x$ (J_x defined as above) such that $\frac{\alpha}{\delta} \leq h + \varepsilon$, $\mu([kb^{-j}, (k+1)b^{-j})) \geq b^{-j\alpha}$ and $|kb^{-j} - x| \leq b\, b^{-j\delta}$ (of course one has $\delta \leq \delta_x$).

Let Kb^{-J} be such an irreducible b-adic number. There exists $y \in [Kb^{-J} - b^{-J\log J}, Kb^{-J} + b^{-J\log J}]$ such that $|\chi_{J,K}(y)| = \|\chi\|_\infty$. This point y also satisfies $|x-y| \leq |x - Kb^{-J}| + |Kb^{-J} - y| \leq b\, b^{-J\delta} + b^{-J\log J}$, which is lower than $Cb^{-J\delta}$ for some constant C when J is large enough. Consequently, if J_y, J'_y and J'_x are defined as above, one has by (4.2)

$$C' b^{-J} \geq b^{-\frac{J_y+1}{\delta}} \geq \frac{1}{C'} b^{-\frac{\delta_x + \varepsilon}{\delta} J'_y}, \text{ or equivalently } J \leq \frac{\delta_x + \varepsilon}{\delta} J'_y + C, \tag{4.4}$$

where C is a constant that does not depend on y or x when J is large enough.

Now one uses the decomposition of $g(y) - g(x)$ into $(A) + (B)$. Hence y is assumed close enough to x so that $J_y \geq J'_x$.

The first term (A) is controlled as above. It remains us to find a lower bound for the term (B).

Let us remark that since Kb^{-J} is irreducible, if $\chi_{j,k}(y) \neq 0$ for some $j < J$ and $0 \leq k < b^j$, then it is necessary that $|kb^{-j} - Kb^{-J}| \leq 2b^{-j\log j}$. In addition, using again that they are both irreducible b-adic numbers of scale smaller than J, one also has $|kb^{-j} - Kb^{-J}| \geq b^{-J}$. One thus obtains that $j \log j \leq J$.

Applying (4.4) gives

$$j \log j \leq \frac{\delta_x + \varepsilon}{\delta} J'_y + C. \tag{4.5}$$

J'_y can be chosen large enough so that (4.5) is possible if and only if $j < J'_y$.

On the other hand, if $j > J$ and $\chi_{j,k}(y) \neq 0$ for some $0 \leq k < b^j$, for the same reasons as above, one has $b^{-j} \leq |kb^{-j} - Kb^{-J}| \leq 2b^{-J\log J}$, which implies $j \geq J\log(J) - \log_b(2)$.

In order to control the term (B), we use $\widetilde{\mathbf{C1}}$ (defined by (1.6)): it yields that $\mu(I_{j,k}) \leq b^{-j\gamma} \leq b^{-\gamma J \log(J)}$ for some $\gamma > 0$. As a consequence, by the same arguments as the ones developed in the beginning of the proof, since J can be chosen large enough so that $J > J'_x$, $\chi_{J,K}(y) \neq 0$ implies that $J \geq J'_y$. Finally,

$$\begin{aligned} |(B)| &\geq \frac{1}{J^2}\mu(I_{J,K})|\chi_{J,K}(y)| - \left| \sum_{j \geq J\log J - \log_b(2)} \frac{1}{j^2} \sum_k \mu(I_{j,k})\chi_{j,k}(y) \right| \\ &\geq \frac{\|\chi\|_\infty}{J^2}\mu(I_{J,K}) - \|\chi\|_\infty b^{-\gamma J\log(J)} \sum_{j \geq J\log J - \log_b(2)} \frac{1}{j^2} \\ &\geq \frac{\|\chi\|_\infty}{J^2}\mu(I_{J,K}) - \frac{\|\chi\|_\infty b^{-\gamma J\log(J)}}{J\log J - \log_b(2)} \geq \frac{\|\chi\|_\infty}{J^2}\mu(I_{J,K}) - \frac{\|\chi\|_\infty b^{-\gamma J\log(J)}}{J\log J - \log_b(2)}. \end{aligned}$$

Since $\mu(I_{J,K}) \geq b^{-(\alpha+\varepsilon)J}$ this yields for some positive constant C

$$\begin{aligned} |(B)| &\geq C\|\chi\|_\infty \frac{b^{-J(\alpha+\varepsilon)}}{J^2} \geq C\|\chi\|_\infty \frac{b^{-(J_y+1)\frac{\alpha+\varepsilon}{\delta}}}{\left(\frac{J_y+1}{\delta}\right)^2} \\ &\geq C\delta^2 \frac{|x-y|^{\frac{\alpha+\varepsilon}{\delta}}}{|\log(|x-y|)|^2} \geq C\frac{|x-y|^{\frac{\alpha+\varepsilon}{\delta}}}{|\log(|x-y|)|^2}, \end{aligned}$$

where (4.4) has been used. Hence,

$$|g(x) - g(y) - P(y)| \geq C\frac{|x-y|^{\frac{\alpha+\varepsilon}{\delta}}}{|\log(|x-y|)|^2} - C|x-y|^{[h]+1}, \tag{4.6}$$

where the constants do not depend on y but on x, α, δ and χ. (4.6) holds for an infinite number of real numbers y, which are converging to x. Thus $h_g(x) \leq \frac{\alpha+\varepsilon}{\delta} \leq h_\nu(x) + \varepsilon + \frac{\varepsilon}{\delta}$, and this is true for every $\varepsilon > 0$. □

The previous arguments do not work when $\delta_x = +\infty$. This is why we do not conclude on the set of Hausdorff dimension 0 consisting of those points x such that $\delta_x = \infty$.

Theorem 1.2 is a consequence of Proposition 4.1 and Theorem 2.9, and of the remark just above.

The reader can check that if there exists a uniform constant $C < 1$ such that for every x, $\sum_{j\geq 1} \mu(I_{j,k_{j,x}}) + \mu(I_{j,k_{j,x}+1}) \leq C$, one can get rid of the $\frac{1}{j^2}$ factor in the definition of the sum of pulses (1.5). This property never holds for the measure ν itself, i.e. when the initial measure μ is taken equal to ν.

Theorem 1.2 possesses an extension similar to the extension Theorem 1.1' of Theorem 1.1. The statement of this extension is left to the reader.

5. Wavelet-based construction derived from ν

In this section, $b=2$ and we work on the dyadic numbers (hence $I_{j,k}=[k2^{-j},(k+1)2^{-j})$.

The approach here is based on the decomposition of functions on orthonormal wavelet bases. Let ψ be a wavelet function in the Schwartz class, such that all its moments of positive orders are null. Under some reconstruction assumptions developped in [16] or [17], the set of functions $\{\psi_{j,k}=\psi(2^j.-k)\}$, where $(j,k)\in\mathbf{Z}^2$, forms an orthogonal basis of $L^2(\mathbf{R})$. Any function $f\in L^2(\mathbf{R})$ can be written

$$f(x)=\sum_{j\in\mathbf{Z}}\sum_{k\in\mathbf{Z}} d_{j,k}\psi_{j,k}(x),$$

where $d_{j,k}$ is the wavelet coefficient of f defined by

$$d_{j,k}:=d_{j,k}(f)=2^j\int_{\mathbf{R}} f(t)\psi_{j,k}(t)dt.$$

ψ can also be chosen with compact support, see [6]. Remark that we take for convenience an L^∞ normalization for the wavelet functions $\psi_{j,k}$.

Let $s_0>0$, $p_0>0$ with $s_0-\frac{1}{p_0}>0$, and let μ be any positive Borel measure. We define the function F_μ

$$F_\mu(x)=\sum_{j\geq 0}\sum_{k\in\mathbf{Z}} d^\mu_{j,k}\psi_{j,k}(x), \tag{5.1}$$

where the wavelet coefficients $d^\mu_{j,k}$ are

$$d^\mu_{j,k}=2^{-j(s_0-\frac{1}{p_0})}|\mu(I_{j,k})|^{\frac{1}{p_0}}. \tag{5.2}$$

In [2], the following theorem is stated

Theorem 5.1. *If μ obeys the multifractal formalism for measures at $\alpha\geq 0$, then $d_{F_\mu}(h)=d_\mu(\alpha)$, where $h=s_0-\frac{1}{p_0}+\frac{\alpha}{p_0}$.*

The function F_μ has its multifractal spectrum simply translated and dilated from the one of μ by the formula $d_{F_\mu}(h)=d_\mu(\alpha)$, where $h=s_0-\frac{1}{p_0}+\frac{\alpha}{p_0}$.

As a consequence, this theorem can be applied to our measures ν and more generally to $\nu_{\gamma,\sigma}$ (see Theorem 2.9'), since they satisfy the multifractal formalism for measures.

Theorem 5.2. *Let μ be a positive Borel measure whose support is $[0,1]$, and assume that* **C1** *holds for μ. Let $\gamma\geq 0$ and $\sigma\geq 1$, $q_{\gamma,\sigma}=\inf\{q\in\mathbf{R}:\tau_\mu(\sigma q)+\gamma q=0\}$, and $h_{\gamma,\sigma}=\tau'_\mu(q^-_{\gamma,\sigma})$.*

(1) If $h_{\gamma,\sigma}>0$ and **C2**$(\frac{h_{\gamma,\sigma}-\gamma}{\sigma})$ *holds, for every $h\in[s_0-\frac{1}{p_0},s_0-\frac{1}{p_0}+\frac{h_{\gamma,\sigma}}{p_0})$, $d_{F_{\nu_{\gamma,\sigma}}}(h)=q_{\gamma,\sigma}p_0(h-(s_0-\frac{1}{p_0}))$.*

(2) If $h\geq h_{\gamma,\sigma}$ and **C3**(h) *holds, then $d_{F_{\nu_{\gamma,\sigma}}}(s_0-\frac{1}{p_0}+\frac{h-\gamma}{\sigma p_0})=\tau^*_\mu(\frac{h-\gamma}{\sigma})$.*

Theorem 5.2 is an easy generalization of Theorem 1.3 taking into account the two parameters γ and σ.

$F_{\nu_{\gamma,\sigma}}$ also satisfies the multifractal formalism for functions as defined in [2], when the conditions **C1-3** are satisfied. In this case, $F_{\nu_{\gamma,\sigma}}$ has a multifractal spectrum which is a dilated and translated version (according to the parameters s_0 and p_0) of the one of $\nu_{\gamma,\sigma}$. For instance, if $\gamma=0$, $\sigma=1$, $s_0=2$ and $p_0=1$, the spectrum of $F_{\nu_{0,1}}$ is the same as the one of ν, but translated by 1 to the right.

This provides a large class of continuous functions with interesting spectra. These spectra are quite different of those of the last sections, because they start at $s_0 - \frac{1}{p_0}$ (instead of $(0,0)$).

One can check that if one takes $\mu = \ell$ (the Lebesgue measure) in the construction of the measure ν, then the corresponding function F_ν^ℓ can be viewed as a perturbation of the so-called *saturating functions* created by S. Jaffard in [13]. Indeed, it is easily shown that the ratios $\rho_{j,k} = \frac{d^\ell_{j,k}}{d'_{j,k}}$ (where $\{d^\ell_{j,k}\}_{j,k}$ is the set of wavelet coefficients of F_ν^ℓ and $\{d'_{j,k}\}_{j,k}$ is the set of wavelet coefficients of the saturating function) are bounded by below and by above by positive constants (the same for all the integers j and k).

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