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Information parameters and large deviations spectrum of discontinuous measures

Abstract

Let ν be a finite Borel measure on $[0,1]^d$. Consider the L^q-spectrum of ν: $\tau_\nu(q) = \liminf_{n\to\infty} -n^{-1}\log_b \sum_{Q\in\mathcal{G}_n} \nu(Q)^q$ ($q \geq 0$), where $\mathcal{G}_n$ is the set of b-adic cubes of generation n (b integer ≥ 2). Let $q_\tau = \inf\{q : \tau_\nu(q) = 0\}$ and $H_\tau = \tau_\nu'(q_\tau^-)$. When ν is a mono-dimensional continuous measure of information dimension D, $(q_\tau, H_\tau) = (1, D)$. When ν is purely discontinuous, its information dimension is $D = 0$, but the pair (q_τ, H_τ) may be non-trivial and contains relevant information on the distribution of ν. Intrinsic characterizations of (q_τ, H_τ) are found, as well as sharp estimates for the large deviations spectrum of ν on $[0, H_\tau]$. We exhibit the differences between the cases $q_\tau = 1$ and $q_\tau \in (0,1)$. We conclude that the large deviations spectrum's properties observed for specific classes of measures are true in general.

1 Introduction

During the last fifteen years, the multifractal behavior of purely discontinuous measures, i.e. constituted only by Dirac masses, has been precisely described for several classes of well-structured objects. Equivalently, the behavior of non-decreasing functions whose derivative is a purely discontinuous measure has also been widely investigated. Examples of such objects are Lévy subordinators [20] and homogeneous sums of Dirac masses studied in [1, 13, 19], Lévy subordinators in multifractal time [8] and more generally heterogeneous sums of Dirac masses governed by a self-similar measure [31, 4, 5]. Nevertheless, few general results are known about the fine structure of purely discontinuous

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measures. For instance, all the possible classical information dimensions vanish for such measures. One of the motivations of the present work is the need for other relevant parameters.

From the multifractal standpoint, a common feature between the above examples is that all their multifractal spectra are linear on a non trivial interval whose left-end point is 0. It is natural to ask whether this property is shared by all or at least a large class of the purely discontinuous measures. This is particularly important for the large deviation spectrum since it is the most numerically tractable spectrum among the several multifractal spectra. Hence, a priori estimates are of great importance for practical purposes. We focus on this spectrum and find that under a weak assumption it is indeed linear on a non trivial interval I whose left-end point is 0 (see Theorem 1.3). The slope of this linear part and the right-end point of I are related to new information parameters deduced from the so-called L^q-spectrum of the measure.

Let us start by recalling the notions of information dimension, multifractal and large deviations spectra. We then expose the achievements of this paper.

1.1 Multifractal spectra and information parameters

In the contexts of fractal sets and dynamical systems, it is usual to describe the geometry and the distribution at small scales of a finite Borel measure ν on $[0,1]^d$ thanks to its (lower and upper) Hausdorff, packing and entropy dimensions. In general these dimensions differ from one another, but when they coincide, they determine without ambiguity the dimension of the measure. This situation arises when there exists $D \in [0,d]$ such that

$$\lim_{r\to 0^+} \frac{\log \nu(B(x,r))}{\log(r)} = D \quad \nu\text{-a.e.} \tag{1}$$

The dimension of ν is equal to D [32, 15], and ν is said to be mono-dimensional.

Let $b \geq 2$ be an integer and let $\mathcal{G}_n$ be the partition of $[0,1]^d$ into b-adic boxes of generation n written as $\prod_{i=1}^d [b^{-n}k_i, b^{-n}(k_i+1))$ with $(k_1, ..., k_d) \in \{0,1,...,b^n-1\}^d$.

Let us introduce on $\mathbb{R}$ the L^q-spectrum of ν

$$\tau_\nu(q) = \liminf_{n\to\infty} -\frac{1}{n}\log_b s_n(q) \quad \text{where } s_n(q) = \sum_{Q\in\mathcal{G}_n,\ \nu(Q)\neq 0} \nu(Q)^q. \tag{2}$$

It is easy to see that τ_ν is a concave function which does not depend on $b \geq 2$ when $q \in \mathbb{R}_+$ or when $q \in \mathbb{R}$ and $\mathrm{Supp}\,(\mu) = [0,1]^d$. Property (1) holds with $D = \tau_\nu'(1)$ for instance as soon as $\tau_\nu'(1)$ exists, and the dimensions mentioned above always lie in $[\tau_\nu'(1^+), \tau_\nu'(1^-)]$ (see Section 2 and [25, 27, 15, 9, 16]).

The behavior of ν at small scales may be more generally geometrically described by the *Hausdorff* and *packing singularity spectra* defined as follows (see [10, 26] and references therein): For $x \in \text{Supp}\,(\nu)$ (the support of ν), the pointwise Hölder exponent of ν at x is defined by

$$h_\nu(x) = \liminf_{r\to 0^+} \frac{\log \nu(B(x,r))}{\log r}, \tag{3}$$

Then, one considers the level sets of the pointwise Hölder exponent of ν

$$E_h^\nu = \{x \in \text{Supp}\,(\nu) : h_\nu(x) = h\} \quad (h \geq 0). \tag{4}$$

Finally, the Hausdorff and packing spectra of ν are respectively

$$d_\nu : h \geq 0 \mapsto \dim E_h^\nu \quad \text{and} \quad D_\nu : h \geq 0 \mapsto \text{Dim}\, E_h^\nu, \tag{5}$$

where dim and Dim stand for the Hausdorff and the packing dimension.

Another description of the distribution of ν is given by the following statistical (rather than geometric) approach. For $\varepsilon > 0$, $h \geq 0$, $n \in \mathbb{N}$, let

$$\mathcal{S}_n^\nu(h,\varepsilon) = \left\{ Q \in \mathcal{G}_n : b^{-n(h+\varepsilon)} \leq \nu(Q) \leq b^{-n(h-\varepsilon)} \right\} \tag{6}$$

The *large deviations spectrum* f_ν of ν is the upper semi-continuous function

$$h \geq 0 \mapsto f_\nu(h) = \lim_{\varepsilon\to 0^+} \limsup_{n\to\infty} \frac{1}{n} \log_b \#\mathcal{S}_n^\nu(h,\varepsilon).$$

The following proposition, which follows from [10, 26, 29, 22, 30] and also Theorem 2.1 hereafter, gathers some properties of these several spectra.

Recall that if g is a function from $\mathbb{R}$ to $\mathbb{R} \cup \{-\infty\}$, its Legendre transform is the mapping $g^* : h \mapsto \inf_{q\in\mathbb{R}}(hq - g(q)) \in \mathbb{R} \cup \{-\infty\}$. If E is a subset of $[0,1]^d$, $\dim E < 0$ means that E is empty.

Proposition 1.1 *Let ν be a finite positive Borel measure on* $[0,1]^d$.

1. *For every $h \geq 0$, $d_\nu(h) \leq f_\nu(h) \leq f_\nu^{**}(h) = \tau_\nu^*(h)$ and $D_\nu(h) \leq \tau_\nu^*(h)$. One says that the multifractal formalism holds at h if $d_\nu(h) = \tau_\nu^*(h)$.*

2. *For every $h \in \{\tau_\nu'(q^+) : q \in \mathbb{R}\} \cup \{\tau_\nu'(q^-) : q \in \mathbb{R}\}$, $f_\nu(h) = f_\nu^{**}(h)$.*

When $D = \tau_\nu'(1)$ exists or (1) holds, the multifractal formalism holds at D and ν is carried by the set E_D^ν ([25]). Examples of continuous measures for which this arises are provided by classes of measures possessing scaling invariance properties (see [23, 24, 21, 28, 10, 14, 18, 16, 3] for a non-exhaustive list).

For these measures one has $D > 0$, thus D is relevant as information dimension since it takes in general different values for two distinct such measures.

On the other hand, it is proved in [11] that in the Baire generic sense, quasi all Borel continuous measures ν on $[0,1]^d$ are concentrated on the set E_D^ν with $D = 0$, and neither $\tau'_\nu(1)$ exists nor (1) holds. This naturally leads to consider the extremal situation when ν is purely discontinuous. In this case, (1) holds with $D = 0$. Hence all the Hausdorff, packing and entropy dimensions equal 0, whatever the behavior of τ_ν at 1 is. Consequently the classical dimension parameters are not relevant in this case.

We thus look for other natural information parameters related to the distribution of a measure. Of course, these parameters must coincide in some sense with the dimension $\tau'_\nu(1)$ when it is defined and positive. We consider

$$q_\tau(\nu) = \inf\{q : \tau_\nu(q) = 0\} \quad \text{and} \quad H_\tau(\nu) = \tau'_\nu(q_\tau(\nu)^-).$$

Let us list some of the properties of these parameters (see also Theorem 2.1):

- If $\dim \mathrm{Supp}\,(\nu) > 0$ and if τ_ν is continuous at 0^+, then one always has $0 < q_\tau(\nu) \leq 1$ and $H_\tau(\nu) \leq d/q_\tau(\nu)$.
- If $\tau'_\nu(1)$ exists and is positive, then $q_\tau(\nu) = 1$ and $H_\tau(\nu) = \tau'_\nu(1)$. Moreover this real number is in this case the only fixed point of f_ν.
- From Proposition 1.1, one gets that $H_\tau(\nu)$ is always the largest solution of the equation $f_\nu(h) = q_\tau(\nu)h$, while $\tau'_\nu(q_\tau(\nu)^+)$ is the smallest solution of the same equation. This implies $f_\nu(H_\tau(\nu)) = \tau^*_\nu(H_\tau(\nu))$.

1.2 Sharper estimates for the large deviations spectrum of ν on $[0, H_\tau(\nu)]$ when ν is a purely discontinuous measure

In the sequel, we focus on purely discontinuous measures of the form

$$\nu = \sum_{k\geq 1} m_k \delta_{x_k} \tag{7}$$

for a sequence of masses $\widetilde{m} = (m_k)_{k\geq 1} \in (\mathbb{R}^+)^{\mathbb{N}^*}$ such that $\sum_k m_k < \infty$ and a sequence of pairwise distinct points $\widetilde{x} = (x_k)_{k\geq 1} \in ([0,1]^d)^{\mathbb{N}^*}$.

Under weak assumptions on the sequences $\widetilde{m}$ and $\widetilde{x}$ (see assumption **(H)** in Definition 1.2 below), one has

$$0 = \tau'_\nu(q_\tau(\nu)^+) < \tau'_\nu(q_\tau(\nu)^-) = H_\tau(\nu).$$

As a consequence of Proposition 1.1 and the third property pointed out above, for such measures, one has $f_\nu(0) = 0$ and $f_\nu(H_\tau) = q_\tau H_\tau$. Moreover, if

$h \in (0, H_\tau)$, then $f_\nu(h) \le q_\tau h$. For all the purely discontinuous measures mentioned in the introduction, $f_\nu(h) = q_\tau h$ on $[0, H_\tau]$. The main purpose of the following work is to understand whether this equality holds true in general.

First, two intrinsic parameters $q_g(\nu)$ and $H_g(\nu)$, depending only on the geometric repartition of $\widetilde{m}$ and $\widetilde{x}$, are proposed in (8) and (10). We investigate them in details. In particular, their relationships with $q_\tau(\nu)$ and $H_\tau(\nu)$ are of great interest and are the subject of a large part of the paper (Sections 3-4).

For every $n \ge 1$, let

$$K_n = \{k : m_k \in [b^{-n}, b^{-(n-1)}]\} \quad \text{and} \quad X_n = \{x_k : k \in K_n\}.$$

X_n contains the locations of the Dirac masses of same order $\sim 2^{-n}$.If $\#K_n = 0$, we set $q(n) = 0$ and $\mathcal{J}(n) = 1$, otherwise if $\#K_n > 0$,

$$q(n) = \frac{\log_b \#K_n}{n} \quad \text{and} \quad \mathcal{J}(n) = \min\Big\{n' : \sup_{Q \in \mathcal{G}_{n'}} \#(Q \cap X_n) \le 1\Big\}.$$

Thus, provided that X_n is not empty, $\#K_n = b^{nq(n)}$, and $\mathcal{J}(n)$ is the first scale which "separates" the elements of X_n. Then let

$$q_g(\nu) = \limsup_{n\to\infty} \, q(n) \quad \text{and} \quad \underline{H}_g(\nu) = \liminf_{j\to\infty} \frac{n}{\mathcal{J}(n)}. \tag{8}$$

For $\alpha > 0$, $n \ge 1$, we set $\mathcal{J}(n,\alpha) = 1$ if $\#K_n = 0$, otherwise we set if $\#K_n > 0$

$$\mathcal{J}(n,\alpha) = \min\left\{n' \in \mathbb{N} : \begin{cases} \text{there is a set } X'_n \subset X_n \text{ of cardinality} \ge b^{n(q(n)-\alpha)} \\ \text{such that } \sup_{Q\in\mathcal{G}_{n'}} \#(Q \cap X'_n) \le 1 \end{cases}\right\}.$$

Provided that $X_n \ne \emptyset$ and α small enough, $\mathcal{J}(n,\alpha)$ is the first scale which separates a large proportion of the elements of X_n. Let us denote by $\mathcal{U}$ the set of positive sequences of real numbers converging to 0. When $q_g(\nu) > 0$, let

$$\forall \varepsilon > 0, \quad H_{g,\varepsilon}(\nu) = \sup_{(\alpha_n)_n \in \mathcal{U}} \limsup_{\substack{n\to\infty, \\ q(n) \ge q_g(\nu)-\varepsilon}} \frac{n}{\mathcal{J}(n,\alpha_n)} \tag{9}$$

$$\text{and} \quad H_g(\nu) = \lim_{\varepsilon\to 0^+} H_{g,\varepsilon}(\nu). \tag{10}$$

Heuristically, asymptotically when $n \to +\infty$, there is no more than $b^{q_g(\nu)n}$ Dirac masses with a weight $\sim b^{-n}$ involved in the sum (7), while $H_g(\nu)$ depends on the proximity between Dirac masses of same order.

Definition 1.2 *The sequences $\widetilde{m}$ and $\widetilde{x}$ are said to satisfy assumption* **(H)** *if $q_g(\nu) > 0$ and $H_g(\nu) > 0$.*

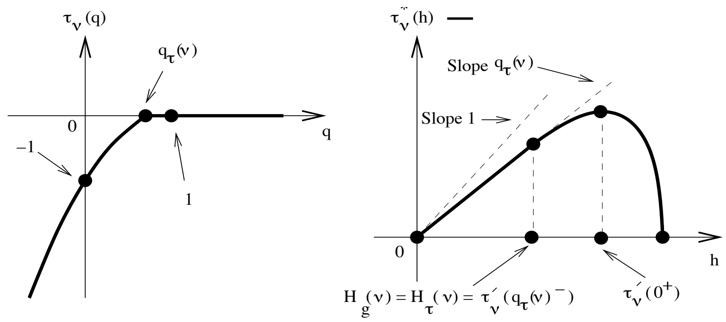

Figure 1: For a discontinuous measure ν with $q_\tau(\nu) < 1$: **Left:** scaling function τ_ν and **Right:** typical Legendre spectrum. The Legendre and large deviations spectra coincide for every $h \leq H_\tau(\nu) = H_g(\nu)$.

Theorem 1.3 *If* **(H)** *is satisfied, then*

1. *If $q_\tau(\nu) \in (0,1)$, then $q_g(\nu) = q_\tau(\nu)$, $H_\tau(\nu) = H_g(\nu)$ and $f_\nu(h) = q_\tau(\nu)h$ for every $h \in [0, H_\tau(\nu)]$.*

2. *If $q_\tau(\nu) = 1$, then $q_g(\nu) = q_\tau(\nu)$ and $H_g(\nu) \leq H_\tau(\nu)$. Moreover, $f_\nu(h) = q_\tau(\nu)h$ for every $h \in [0, H_g(\nu)]$ and $f_\nu(H_\tau(\nu)) = q_\tau(\nu)H_\tau(\nu)$ (i.e. the large deviations spectrum may exhibit a gap between $H_g(\nu)$ and $H_\tau(\nu)$).*

3. *When $q_\tau(\nu) = 1$, the exponent $H_g(\nu)$ is optimal in the following sense:*

   (a) For every exponents $0 < h_0 \leq h_1 \leq d$, there exists a measure ν of the form (7) such that $q_\tau(\nu) = 1$, $H_g(\nu) = h_0$, $H_\tau(\nu) = h_1$ and $f_\nu(h) = h$ for every $h \in [0, h_1]$.

   (b) For every exponents $0 < h_0 < h_1 \leq d$, there is a measure ν of the form (7) such that $q_\tau(\nu) = 1$, $H_g(\nu) = h_0$, $H_\tau(\nu) = h_1$ and $f_\nu(h) < h$ on (h_0, h_1).

4. *Assume that for all $\varepsilon > 0$ there is an increasing sequence of integers $(n_j)_{j\geq 1}$ such that $q(n_j)$ converges to 1 and*

$$\dim\Big(\limsup_{j\to\infty} \bigcup_{k\in K_{n_j}} B(x_k, b^{-n_j(H_\tau(\nu)^{-1}-\varepsilon)})\Big) \geq H_\tau(\nu).$$

   Then $H_g(\nu) = H_\tau(\nu)$.

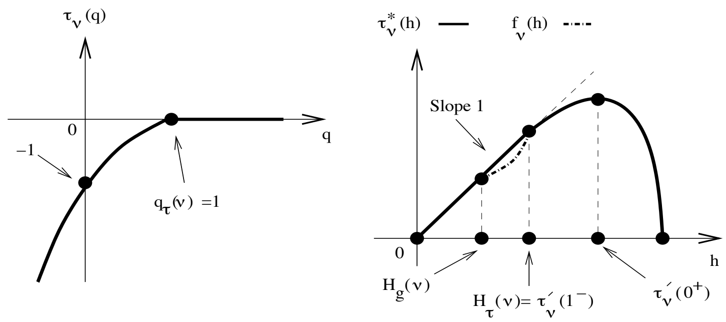

Figure 2: For a discontinuous measure ν with $q_\tau(\nu) = 1$: **Left:** scaling function τ_ν and **Right:** typical Legendre and large deviations spectra. They coincide when $h \le H_g(\nu)$ and when $h = H_\tau(\nu)$.

When **(H)** is satisfied, the relation between $(q_g(\nu), H_g(\nu))$ and $(q_\tau(\nu), H_\tau(\nu))$ is pointed out, and an intrinsic interpretation of $(q_\tau(\nu), H_\tau(\nu))$ is exhibited. Moreover, the large deviations spectrum of any measure ν (7) satisfying **(H)** always starts with a straight line, of slope $q_\tau(\nu)$. The reader can check that assumption **(H)** holds true and $H_g(\nu) = H_\tau(\nu)$ for most of the measures mentioned in the very beginning of this section (see the comments in Section 1.3). Hence Theorem 1.3 provides us with a global standpoint which allows us to recover the linear increasing part of the large deviations spectrum.

The case where $q_g(\nu) > 0$ but $H_g(\nu) = 0$ is also interesting (though exceptional) and arises for instance if $q_\tau(\nu) > 0$ and $H_\tau(\nu) = 0$. Examples of such measures are constructed in [4]. We dot not consider this situation hereafter.

1.3 Remarks and related works

• It is worth noticing that $q_g(\nu)$, $H_g(\nu)$, $q_\tau(\nu)$ and $H_\tau(\nu)$ do no depend on the choice of the integer basis b (while f_ν clearly depends on b in general). We have thus established that, under **(H)**, f_ν is always linear with slope $q_\tau(\nu)$ on the range of exponents $h \in [0, H_g(\nu)]$ for every choice of $b \ge 2$.

• There is a strong asymmetry between the cases $q_g(\nu) = 1$ and $q_g(\nu) \in (0,1)$. In this latter case, the large deviations spectrum is totally determined by $H_g(\nu) = H_\tau(\nu)$ on its linear part starting at $(0,0)$. In the case $q_g(\nu) = q_\tau(\nu) = 1$, it is only known for $h \in [0, H_g(\nu)]$ and at $H_\tau(\nu)$ (which may be $> H_g(\nu)$). Nevertheless, when $q_g(\nu) = 1$, item (4) of Theorem 1.3 gives a sufficient condition to have $H_\tau(\nu) = H_g(\nu)$, and the large deviations spectrum increases along a straight line with slope 1 until it reaches $H_\tau(\nu)$. This condition imposes an homogeneous repartition of the Dirac masses of same intensity.

• The reader shall keep in mind that no comparable result can hold if the large deviations spectrum is replaced by the Hausdorff multifractal spectrum.

• For the examples of homogeneous and heterogeneous sums of discontinuous measures studied in [1, 20, 13, 31, 30, 4, 5, 8], it is not difficult to verify, thanks to Theorem 1.3(4), that when $H_\tau(\nu) > 0$, $H_g(\nu) = H_\tau(\nu)$. For instance, the derivative ν_β of a stable Lévy subordinator L_β of index $\beta \in (0,1)$ satisfies $H_g(\nu) = H_\tau(\nu) = 1/\beta$ and $q_\tau(\nu_\beta) = \beta$ (see [20]).

• The following remarks complete Theorem 1.3. Under assumption **(H)**, the parameter $q_\tau(\nu)$ depends on $\widetilde{m}$ and not on the locations $\widetilde{x}$. In a forthcoming work we study, the value of $q_\tau(\nu)$ being given, the influence on $H_\tau(\nu)$ of the locations of the Dirac masses $(x_k)_{k\geq 1}$.

• Finally, in the companion paper [6], we illustrate the important role played by the information parameters $(q_\tau(\nu), H_\tau(\nu))$ for the Hausdorff spectrum. An interesting example is provided by a class of discontinuous measures ν_b introduced in [4], whose atoms are located at b-adic numbers of $[0,1]$. For such measures ν_b, $H_g(\nu_b) = H_\tau(\nu_b)$ even when $q_\tau(\nu_b) = 1$. Due to Proposition 3.3 of the present paper, a natural procedure is to apply a threshold to ν_b by keeping only the masses which contribute to the fact that $f_{\nu_b}(H_\tau(\nu_b)) = q_\tau(\nu_b)H_\tau(\nu_b)$. This yields a second measure ν_b^t. It is shown that ν_b and ν_b^t have the same multifractal properties on $[0, H_\tau(\nu_b)]$ in the sense that their Hausdorff, large deviations and Legendre spectra coincide. Moreover, when $q_\tau(\nu_b) = 1$, these spectra also coincide at the exponents $h > H_\tau(\nu_b)$. This striking result confirms that valuable information on the local behavior of ν are somehow stored in the masses which are detected by $(q_\tau(\nu), H_\tau(\nu))$.

2 Universal bounds for the large deviations spectrum

For $j \geq 1$ and $x \in (0,1)^d$, $Q_j(x)$ is the unique b-adic cube of scale $j \geq 1$ containing x, and for every $\eta \in \{-1,0,1\}^d$, $Q_j^{(\eta)}(x) = Q_j(x) + b^{-j}\eta$. In the following, $|B|$ always denotes the diameter of the set B. Eventually, for the rest of the paper, the convention $\log(0) = -\infty$ is adopted.

2.1 Links between $(q_\tau(\nu), H_\tau(\nu))$ and the large deviations spectrum

Next result goes slightly beyond Proposition 1.1(1) and it resumes some comments ending Section 1.1.

Theorem 2.1 *Let ν be a finite Borel measure on $[0,1]^d$. Suppose that τ_ν is continuous at 0^+ and* $\dim \mathrm{Supp}\,(\nu) > 0$. *Set $H_\tau^+(\nu) = \tau_\nu'(q_\tau(\nu)^+)$. One has*

1. $q_\tau(\nu) > 0$.
2. *For every* $h \geq 0$, $d_\nu(h) \leq f_\nu(h) \leq \tau_\nu^*(h)$. *Moreover* $\tau_\nu^*(h) = q_\tau(\nu)h$ *if* $h \in [\tau_\nu'(q_\tau(\nu)^+), H_\tau(\nu)]$ *and* $\tau_\nu^*(h) < q_\tau(\nu)h$ *otherwise.*
3. $H_\tau^+(\nu) = \min\{h \geq 0 : f_\nu(h) = q_\tau(\nu)h\} = \min\{h \geq 0 : \tau_\nu^*(h) = q_\tau(\nu)h\}$.
4. $H_\tau(\nu) = \max\{h \geq 0 : f_\nu(h) = q_\tau(\nu)h\} = \max\{h \geq 0 : \tau_\nu^*(h) = q_\tau(\nu)h\}$.
5. *If* $E_0^\nu \neq \emptyset$, *then* $\tau_\nu'(q_\tau(\nu)^+) = 0$.

Remark 2.2 *Notice that, if* $E_0^\nu \neq \emptyset$ *and* $q_\tau(\nu) > 0$, *then for every* $h \in [0, H_\tau(\nu)]$, $\tau_\nu^*(h) = q_\tau h$, *while this may not be the case for* f_ν: *There may exist* $0 < h < H_\tau(\nu)$ *such that* $f_\nu^*(h) < q_\tau h$ *(see Theorem 1.3(3)(b) for instance).*

Remark 2.3 *Let* $\widehat{E}_h^\nu = \left\{x : \lim_{j\to\infty} \frac{\log \nu(Q_j(x))}{-j\log b} = h\right\}$. *In Theorem 2.1(2), the inequality* $d_\nu(h) \leq f_\nu(h)$ *is a refinement of the well-known inequality* $\dim \widehat{E}_h^\nu \leq f(h)$ *[10, 29]. For classes of continuous measures possessing some self-similarity property, one often has* $\dim \widehat{E}_h^\nu = f_\nu(h)$ *for all* h *such that* $f_\nu(h) \geq 0$.

We include the inequality $d_\nu(h) \leq f_\nu(h)$ *in the statement because for purely discontinuous measures studied in [31, 20, 13, 4], or the derivative of a generic increasing continuous function [11], the set* $\widehat{E}_h^\nu$ *is empty for* $h \in (0, H_\tau(\nu))$ *while* $\dim E_h^\nu = \tau_\nu^*(h)$ *and thus* $d_\nu(h) = f_\nu(h)$. *This emphasizes the fact that in general sets like* E_h^ν *or* $\left\{x : \liminf_{j\to\infty} \frac{\log \nu(Q_j(x))}{-j\log b} = h\right\}$ *must be used rather than* $\widehat{E}_h^\nu$ *to describe the local behavior of* ν. *Then* $f_\nu(h)$ *provides a convenient upper bound estimate for* $\dim E_h^\nu$ *rather than for* $\dim \widehat{E}_h^\nu$.

Proof: We simply write $(q_\tau, H_\tau) = (q_\tau(\nu), H_\tau(\nu))$.

1. Assume that $\dim \mathrm{Supp}\,(\nu) > 0$ and τ_ν is continuous at 0^+. One clearly has that $\dim \mathrm{Supp}\,(\nu) \leq -\tau_\nu(0)$, thus $0 < -\tau_\nu(0)$. The continuity and monotonicity of τ_ν at 0^+ and the fact that $\tau_\nu(1)$ always equals 0 yield the result.

2. We shall need the following simple lemma.

Lemma 2.4 *Let* ν *be a positive Borel measure on* $[0,1]$ *and* $x \in (0,1)$. *One has* $h_\nu(x) = \min_{\eta\in\{-1,0,1\}^d} h_\nu^{(\eta)}(x)$, *where* $h_\nu^{(\eta)}(x) = \liminf_{j\to\infty} \frac{\log \nu(Q_j^{(\eta)}(x))}{-j\log b}$.

For every $h \geq 0$, the fact that $f_\nu(h) \leq \tau_\nu^*(h) \leq q_\tau h$ is a well-known property already mentioned in the introduction, and the comparison between $\tau_\nu^*(h)$ and $q_\tau(\nu)h$ follows from the definition of τ_ν^*. We only prove the inequality $d_\nu(h) \leq f_\nu(h)$ for $h \geq 0$ such that $d_\nu(h) > 0$. Indeed, the proof of this inequality (using the liminf in the definition of the exponent), though very fast to obtain, has never been written entirely, according to our best knowledge.

Let $\varepsilon>0$. By definition of $f_\nu(h)$, for every n large enough, $\#\mathcal{S}_n^\nu(h,\varepsilon)\le b^{n(f_\nu(h)+\varepsilon)}$ (see (6)). Let us consider the limsup set

$$K_\nu(h,\varepsilon)=\bigcap_{N\ge 1}\ \bigcup_{n\ge N}\ \bigcup_{Q\in\mathcal{S}_n^\nu(h,\varepsilon)}\ \bigcup_{\eta\in\{-1,0,1\}^d} Q^{(\eta)}.$$

(Recall that when Q is a b-adic cube of generation n, $Q^{(\eta)}=Q+\eta b^{-j}$.) Let now $x\in E_h^\nu$. By Lemma 2.4, there exists a sequence $(n_j^x)_{j\ge1}$ such that for every $j\ge 1$, there is $\eta\in\{-1,0,1\}^d$ such that $b^{-n(h+\varepsilon)}\le\nu(Q^{(\eta)}_{n_j^x}(x))\le b^{-n(h-\varepsilon)}$. As a consequence, $x\in K_\nu(h,\varepsilon)$, thus $E_h^\nu\subset K_\nu(h,\varepsilon)$.

We now find an upper bound for $\dim K_\nu(h,\varepsilon)$. Let $t>f_\nu(h)+\varepsilon$, and let us estimate the t-Hausdorff measure of the set $K_\nu(h,\varepsilon)$. For $N\ge1$, the union $\bigcup_{n\ge N}\ \bigcup_{Q\in\mathcal{S}_n^\nu(h,\varepsilon)}\ \bigcup_{\eta\in\{-1,0,1\}^d}Q^{(\eta)}$ forms a covering of $K_\nu(h,\varepsilon)$. Then,

$$\begin{aligned}\sum_{n\ge N}\ \sum_{Q\in\mathcal{S}_n^\nu(h,\varepsilon)}\ \sum_{\eta\in\{-1,0,1\}^d}|Q^{(\eta)}|^t &\le \sum_{n\ge N}\ \sum_{Q\in\mathcal{S}_n^\nu(h,\varepsilon)} 3^d b^{-nt}\\ &\le 3^d\sum_{n\ge N} b^{-nt}b^{n(f_\nu(h)+\varepsilon)}.\end{aligned}$$

This last sum converges, since $t>f_\nu(h)+\varepsilon$, and its value goes to zero when N goes to infinity. As a consequence, the t-Hausdorff measure of $K_\nu(h,\varepsilon)$ equals zero, and $d_\nu(h)\le\dim K_\nu(h,\varepsilon)\le f_\nu(h)+\varepsilon$. Letting $\varepsilon\to0$ yields the result.

3. and 4. As we already noticed, these properties are consequences of Proposition 1.1 and the definition of the Legendre transform.

5. One always has $\tau_\nu(q_\tau)=0$. If $q_\tau<1$, since τ_ν is concave and positive when $q\ge1$, then $\tau_\nu(q)=0$ for every $q\ge q_\tau$, and in particular it is zero at q_τ^+.

Assume that $q_\tau=1$. Let $x\in E_0^\nu$. Then fix $\varepsilon>0$ and $(r_j)_{j\ge1}$ a sequence decreasing to 0 such that $\nu(B(x,r_j))\ge(r_j)^\varepsilon$ for all $j\ge1$. Let n_j be the unique integer such that $b^{-n_j}\le 2r_j\le b^{-n_j+1}$. One of the (at most) b^d b-adic cubes of generation n_j intersecting $B(x,r_j)$, say Q_j, satisfies $\nu(Q_j)\ge r_j^\varepsilon/b^d\ge b^{-n_j\varepsilon}b^{-d}$. Let $q>1$. Remembering (2), one gets $s_{n_j}(q)\ge b^{-n_j\varepsilon}b^{-d}$. Taking the liminf yields $\tau_\nu(q)\le\varepsilon$, which holds $\forall q\ge1$ and $\varepsilon>0$. Hence the result. ∎

2.2 Additional definitions and large deviations bounds

Let ν be a positive Borel measure on $[0,1]^d$. Before establishing Theorem 1.3, some definitions and estimates of f_ν and related quantities are needed. For $x\in(0,1)^d$, recall the definitions (3) of the Hölder exponent at x and of the corresponding level sets E_h^ν (4), for $h\ge0$. We mention the following result.

Definition 2.5 *Let ν be a positive Borel measure on $[0,1]^d$. For $h \geq 0$ and $n \geq 1$ let us introduce*

$$\underline{N}_\nu(h,n) = \#\left\{Q \in \mathcal{G}_n : b^{-nh} \leq \nu(Q)\right\}, \quad \underline{f}_\nu(h) = \limsup_{n\to\infty} \ n^{-1}\log_b \underline{N}_\nu(h,n).$$

Hence $\underline{f}_\nu(h)$ is related to the asymptotic number of b-adic cubes Q of generation n such $\nu(Q) \geq b^{-nh}$.

Using [10] and the definition of the Legendre transform, one gets the useful following properties (some of them were recalled in Proposition 1.1).

Proposition 2.6 *Let ν be a positive Borel measure on $[0,1]^d$ with $q_\tau(\nu) > 0$.*

1. *For every exponent $h > \tau_\nu'(0^+)$, one has $f_\nu(h) \leq \overline{f}_\nu(h) \leq \tau_\nu^*(h)$ and $\underline{f}_\nu(h) \leq q_\tau(\nu)\tau_\nu'(0^+) < q_\tau(\nu)h$.*
2. *If $h \in [0, \tau_\nu'(0^+)]$, then $f_\nu(h) \leq \underline{f}_\nu(h) \leq \tau_\nu^*(h) \leq q_\tau(\nu)h$.* *Moreover, if $\underline{f}_\nu(h) = q_\tau(\nu)h$, then $f_\nu(h) = q_\tau(\nu)h$.*

3 Theorem 1.3(1-2): Characterization of $(q_\tau(\nu), H_\tau(\nu))$

Let $\widetilde{m} = (m_k)_{k\in\mathbb{N}}$ be a sequence of positive numbers such that $\sum_{k\in\mathbb{N}} m_k < \infty$ and $\widetilde{x} = (x_k)_{k\in\mathbb{N}} \in \left([0,1]^d\right)^{\mathbb{N}}$ a sequence of points, and consider the purely discontinuous measure ν defined by (7).

Let us begin with a Proposition relating the quantities $q_g(\nu)$ and $H_g(\nu)$.

Proposition 3.1 *Let $\widetilde{m} = (m_k)_{k\in\mathbb{N}}$ be a sequence of positive numbers such that $\sum_{k\in\mathbb{N}} m_k < \infty$ and $\widetilde{x} = (x_k)_{k\in\mathbb{N}}$ a sequence of pairwise distinct points in $[0,1]^d$. Let $\nu = \sum_{k\in\mathbb{N}} m_k\ \delta_{x_k}$. If $q_g(\nu) > 0$, then $0 \leq H_g(\nu) \leq d/q_g(\nu)$.*

Proof: Let $(n_j)_{j\geq 1}$ be an increasing sequence of integers and $(\alpha_j)_{j\geq 1}$ be a non-increasing positive sequence going down to 0 such that $\lim_{j\to+\infty} \frac{\log_b \#X_{n_j}(\alpha_j)}{n_j} = q_g(\nu)$ and $\lim_{j\to+\infty} n_j/\mathcal{J}(n_j,\alpha_j) = H_g(\nu)$.

Let $\varepsilon > 0$, small enough so that $q_g(\nu) - 2\varepsilon > 0$. There is $j_\varepsilon \geq 0$ such that $j \geq j_\varepsilon$ implies $\left|\frac{\log_b \#X_{n_j}(\alpha_j)}{n_j} - q_g(\nu)\right| \leq \varepsilon$ and $|n_j/\mathcal{J}(n_j,\alpha_j) - H_g(\nu)| \leq \varepsilon$.

At scale n_j (where $j \geq j_\varepsilon$), one has $b^{n_j(q_g(\nu)-\varepsilon)} \leq \#X_{n_j}(\alpha_j) \leq b^{n_j(q_g(\nu)+\varepsilon)}$. Let $n \leq n_j((q_g(\nu)-2\varepsilon)/d)$. The cardinality of $\mathcal{G}_n$ is $b^{nd} \leq b^{n_j(q_g(\nu)-2\varepsilon)}$. Hence, at least one b-adic cube of $\mathcal{G}_n$ contains two points of $X_{n_j}(\alpha_j)$. Consequently, by definition one has $\mathcal{J}(n_j,\alpha_j) \geq n_j((q_g(\nu)-2\varepsilon)/d)$, and $n_j/\mathcal{J}(n_j,\alpha_j) \leq d/(q_g(\nu)-2\varepsilon)$.

By letting j go to infinity, one obtains $H_g(\nu) \leq d/(q_g(\nu)-4\varepsilon)$, and the result follows by letting ε go to zero. ∎

3.1 An intrinsic characterization of $q_\tau(\nu)$

Theorem 3.2 *Let $\widetilde{m}=(m_k)_{k\in\mathbb{N}}$ be a sequence of positive numbers such that $\sum_{k\in\mathbb{N}} m_k<\infty$ and $\widetilde{x}=(x_k)_{k\in\mathbb{N}}$ a sequence of pairwise distinct points in $[0,1]^d$. Let $\nu=\sum_{k\in\mathbb{N}} m_k\ \delta_{x_k}$.*

(1) One has $q_\tau(\nu)\le q_g(\nu)\le 1$. (2) If **(H)** *holds, then $q_g(\nu)=q_\tau(\nu)$.*

Proof: • (1) One has $q_g(\nu)\le 1$ because $\widetilde{m}$ is summable. If $q_g(\nu)=1$, then $q_\tau(\nu)\le 1=q_g(\nu)$ and the result is proved. We thus assume that $q_g(\nu)<1$, and we prove that $q_\tau(\nu)\le q_g(\nu)$.

Let $\varepsilon>0$ be such that $q_g(\nu)+2\varepsilon\le 1$ and $n_\varepsilon\ge 1$ such that $\#K_n\le b^{n(q_g(\nu)+\varepsilon)}$ for $n\ge n_\varepsilon$. Then, using the sub-additivity of the mapping $t\mapsto t^{q_g(\nu)+2\varepsilon}$ on $\mathbb{R}_+$ we see that for all $n\ge 1$

$$\begin{aligned} s_n(q_g(\nu)+2\varepsilon) &\le \sum_{k\in\mathbb{N}} m_k^{q_g(\nu)+2\varepsilon}\\ &\le \sum_{1\le n'<n_\varepsilon}\sum_{k\in K_{n'}} m_k^{q_g(\nu)+2\varepsilon}+\sum_{n'\ge n_\varepsilon}\sum_{k\in K_{n'}} m_k^{q_g(\nu)+2\varepsilon}\\ &\le \sum_{1\le n'<n_\varepsilon}\sum_{k\in K_{n'}} m_k^{q_g(\nu)+2\varepsilon}+\sum_{n'\ge n_\varepsilon}(\#K_{n'})b^{-(n'-1)(q_g(\nu)+2\varepsilon)}\\ &\le \sum_{1\le n'<n_\varepsilon}\sum_{k\in K_{n'}} m_k^{q_g(\nu)+2\varepsilon}+\sum_{n'\ge n_\varepsilon} b^{n'(q_g(\nu)+\varepsilon)}b^{-(n'-1)(q_g(\nu)+2\varepsilon)}. \end{aligned}$$

The first term of the right hand side of the last inequality does not depend on n, and the second one converges since upper bounded by $b^{q_g(\nu)+\varepsilon}\sum_{n'\ge n_\varepsilon} b^{-n'\varepsilon}$. Hence, $s_n(q_g(\nu)+2\varepsilon)$ is bounded independently of n. This yields $\tau_\nu(q_g(\nu)+2\varepsilon)=0$, and so $q_\tau(\nu)\le q_g(\nu)+2\varepsilon$. This is true $\forall\varepsilon>0$, hence the result.

• Let us now prove that $q_g(\nu)=q_\tau(\nu)$ under the assumption that $q_g(\nu)>0$ and $\underline{H}_g(\nu)>0$ (see (8)). The fact that $q_g(\nu)=q_\tau(\nu)$ under the weaker assumption **(H)** follows from the proof of Proposition 3.3. Let $\varepsilon\in(0,q_g(\nu))$.

Let $(n_j)_{j\ge 1}$ be an increasing integers sequence converging to ∞, and $(\varepsilon_j)_{j\ge1}$ a positive sequence converging to 0 such that $\forall j\ge 1$, $b^{n_j(q_g(\nu)-\varepsilon_j)}\le \#K_{n_j}$.

For every $j\ge 1$, recall that $p_j=\mathcal{J}(n_j)$ is the first scale which separates the elements of K_{n_j}. When j is large enough so that $\varepsilon_j\le\varepsilon/4$, one has

$$\begin{aligned} s_{p_j}(q_g(\nu)-\varepsilon) &\ge \sum_{k\in K_{n_j}} m_k^{q_g(\nu)-\varepsilon}\\ &\ge (\#K_{n_j})b^{-n_j(q_g(\nu)-\varepsilon)}\ge b^{n_j(q_g(\nu)-\varepsilon_j)}b^{-n_j(q_g(\nu)-\varepsilon)}, \end{aligned}$$

which equals $b^{n_j(\varepsilon-\varepsilon_j)}$. Thus $-\log(s_{p_j}(q_g(\nu)-\varepsilon))/p_j\le-(\varepsilon-\varepsilon_j)n_j/p_j$. By assumption, $\liminf_{j\to+\infty} n_j/p_j\ge\underline{H}_g(\nu)>0$. Taking the liminf yields $\tau_\nu(q_g(\nu)-\varepsilon)\le-\varepsilon\underline{H}_g(\nu)<0$. Hence $q_\tau(\nu)>q_g(\nu)-\varepsilon$, for every $\varepsilon>0$. ∎

3.2 Preliminary work for the large deviations spectrum

Proposition 3.3 *Let $\widetilde{m} = (m_k)_{k\in\mathbb{N}}$ be a sequence of positive numbers such that $\sum_{k\in\mathbb{N}} m_k < 1$ and $\widetilde{x} = (x_k)_{k\in\mathbb{N}}$ a sequence of points in $[0,1]^d$. Let $\nu = \sum_{k\in\mathbb{N}} m_k \ \delta_{x_k}$.*

Suppose that **(H)** *is satisfied (this implies that $q_c(\nu) = q_g(\nu)$).*

Let $h_0 = H_\tau(\nu)$ if $q_\tau(\nu) \in (0,1)$ and $h_0 = H_g(\nu)$ if $q_\tau(\nu) = 1$.

There exist a sequence $(p_j)_{j\geq 1}$ of integers going to ∞, a positive sequence $(\varepsilon_j)_{j\geq 1}$ going to 0 , and a sequence of sets of b-adic boxes $(B_j)_{j\geq 1}$ such that

1. $\lim_{j\to+\infty} \frac{\log_b \#\mathcal{S}^\nu_{p_j}(h_0,\varepsilon_j)}{p_j} = q_\tau(\nu)h_0$;
2. *For every $j \geq 1$, $B_j \subset \mathcal{S}^\nu_{p_j}(h_0,\varepsilon_j)$;*
3. $\lim_{j\to\infty} \frac{\log_b(\#B_j)}{p_j} = q_\tau(\nu)h_0$;
4. *For every $Q \in B_j$, there exists $k \in \mathbb{N}$ such that if $x_Q := x_k \in Q$, $\lim_{j\to\infty} \sup_{Q\in B_j} \left| \frac{\log_b m_Q}{-p_j} - h_0 \right| = 0$, where $m_Q = m_k$ if $x_Q = x_k$.*

Proposition 3.3 plays a crucial role in proving Theorem 1.3. It asserts that when $q_\tau(\nu) < 1$ there exists a sequence ε_n going to 0 and infinitely many n such that $\mathcal{S}^\nu_n(H_\tau(\nu),\varepsilon_n) \approx b^{nq_\tau(\nu)H_\tau(\nu)}$, a substantial proportion of the cubes Q of generation n such that $\nu(Q) \approx b^{-nH_\tau(\nu)}$ contains a point x_k such that its associated mass satisfies $m_k \approx b^{-nH_\tau(\nu)}$. The same general fact holds true only with $H_g(\nu)$ instead of $H_\tau(\nu)$ if $q_\tau(\nu) = 1$.

Proof: **(i)** We first prove that the result holds true with $h_0 = H_g(\nu)$ whatever the value of $q_\tau(\nu)$ is. We then establish property (1) of the statement with $q_g(\nu)$ instead of $q_\tau(\nu)$. Since $q_g(\nu) \geq q_\tau(\nu)$ (Theorem 3.2(1)), this implies $q_g(\nu) = q_\tau(\nu)$ by Proposition 2.6.

By definition of $H_g(\nu)$, there is an increasing sequence of integers $(n_j)_{j\geq 1}$ and a positive sequence $(\alpha_j)_{j\geq 1}$ going to 0 such that $\lim_{j\to+\infty} \frac{\log_b \#X_{n_j}(\alpha_j)}{n_j} = q_g(\nu)$ and simultaneously $\lim_{j\to+\infty} \frac{n_j}{\mathcal{J}(n_j,\alpha_j)} = H_g(\nu)$.

Let $\varepsilon_0 \in \big(0, \min\big(q_g(\nu), H_g(\nu)\big)\big)$, and take $(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4) \in (0,\varepsilon_0)^4$. One has, for some integer j_1 large enough,

- for every $j \geq j_1$, $\left| \frac{\log_b \#X_{n_j}(\alpha_j)}{n_j} - q_g(\nu) \right| \leq \varepsilon_1$ and $\left| \frac{n_j}{\mathcal{J}(n_j,\alpha_j)} - H_g(\nu) \right| \leq \varepsilon_2$,
- for every $n \geq n_{j_1}$, $\frac{\log_b \#\underline{N}_\nu(H_g(\nu)-\varepsilon_4, n)}{n} \leq q_\tau(\nu)(H_g(\nu) - \varepsilon_4) + \varepsilon_3$.

The second point holds true due to Proposition 2.6(1) and (2).

Let $p_1 = \mathcal{J}(n_{j_1})$. Consider $\mathcal{G}_{p_1}$. By construction of p_1, there are $\#X_{n_{j_1}}(\alpha_{j_1})$ b-adic boxes Q of $\mathcal{G}_{p_1}$ that contain a point x_Q such that $x_Q = x_k$ for some $x_k \in X_{n_{j_1}}(\alpha_{j_1})$ (with the associated mass denoted m_Q). Each of these b-adic boxes Q satisfies $\nu(Q) \geq m_Q \geq b^{-n_{j_1}}$, which is greater than $b^{-p_1(H_g(\nu)+\varepsilon_2)}$. One can also write that $\#X_{n_{j_1}}(\alpha_{j_1}) \geq b^{n_{j_1}(q_\tau(\nu)-\varepsilon_1)} \geq b^{p_1(H_g(\nu)-\varepsilon_2)(q_g(\nu)-\varepsilon_1)} = b^{p_1(H_g(\nu)q_g(\nu)-\varepsilon_5)}$, with $\varepsilon_5 = |\varepsilon_1\varepsilon_2 - \varepsilon_1 H_g(\nu)) - \varepsilon_2 q_g(\nu)|$.

By the second assumption above on j_1, one knows that the cardinality of the set of cubes $Q \in \mathcal{G}_{p_1}$ such that $\nu(Q) \geq b^{-p_1(H_g(\nu)-\varepsilon_4)}$ is less than $b^{p_1(q_\tau(\nu)H_g(\nu)-\varepsilon_6)}$, with $\varepsilon_6 = \varepsilon_4 q_\tau(\nu) - \varepsilon_3$.

The reader easily verifies that $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \in (0, \varepsilon_0)^4$ can be chosen so that $0 \leq \varepsilon_5 < \varepsilon_6$. In this case, since $q_g(\nu) \geq q_\tau(\nu)$, there exists $\eta_1 > 0$ such that there are at least $b^{p_1(H_g(\nu)q_g(\nu)-\varepsilon_5)} - b^{p_1(q_\tau(\nu)H_g(\nu)-\varepsilon_6)} = b^{p_1(H_g(\nu)q_\tau(\nu)-\eta_1)}$ b-adic boxes of generation p_1 such that $b^{-p_1(H_g(\nu)+\varepsilon_2)} \leq \nu(Q) \leq b^{-p_1(H_g(\nu)-\varepsilon_4)}$.

Let us set $\gamma_1 = \max(\varepsilon_2, \varepsilon_4) \leq \varepsilon_0$. We proved that there is a set of b-adic boxes B_{p_1}, of cardinality $b^{p_1(H_g(\nu)q_g(\nu)-\eta_1)}$ such that for every $Q \in B_{p_1}$,

- Q is also included in $S_{p_1}(H_g(\nu), \gamma_1)$,
- there is a real number x_Q in Q such that $x_Q = x_k$ for some $k \in \mathbb{N}$ such that k verifies $\left|\frac{\log_b m_k}{p_1} - H_g(\nu)\right| \leq \gamma_1$.

It is obvious that there exists a constant C depending only on $q_g(\nu)$ and $H_g(\nu)$ such that η_1 can be chosen so that $0 \leq \max(\gamma_1, \eta_1) \leq C\varepsilon_0$.

We then construct the sequence $(p_j)_{j\geq 1}$ by induction, by iterating the same procedure at each step $i \geq 1$ (where the construction at step i is achieved using $\varepsilon_i = \min(\varepsilon_{i-1}, \eta_{i-1})/(2C)$ instead of ε_0).

(ii) We now consider the case when $q_\tau(\nu) \in (0,1)$ and $h_0 = H_\tau(\nu)$. The result is a consequence of the following lemma.

Lemma 3.4 *Suppose that $q_\tau(\nu) = q_g(\nu) = q_\tau \in (0,1)$ and $H_\tau(\nu) = H_\tau > 0$. Let $C_2 > \frac{2q_\tau}{1-q_\tau}$ and $\widetilde{C}_2 = C_2(1-q_\tau) - q_\tau$. If $\varepsilon > 0$ is small enough, then there exists a sequence $(j_p)_{p\geq 1}$ going to ∞, a sequence of sets of b-adic boxes $(B_p)_{p\geq 1}$ and a constant $C(\varepsilon) \in (0, \widetilde{C}_2)$ such that for all $p \geq 1$:*

1. *$B_p \subset \mathcal{S}^\nu_{j_p}(H_\tau, C(\varepsilon)\varepsilon)$ and $\#B_p \geq b^{j_p q_\tau H_\tau(1-\varepsilon)}$.*
2. *for every $Q \in B_p$, there exists $k \geq 1$ such that $x_k \in B_p$ and $b^{-j_p H_\tau(1+C_2\varepsilon)} \leq m_k \leq b^{-j_p H_\tau(1-C(\varepsilon)\varepsilon)}$.*

Proof: Recall Theorem 2.1(5). Fix $\varepsilon \in (0,1)$. Then let $\eta_0 > 0$ such that for all $\eta \in (0, \eta_0)$, there exists a sequence $(j_p)_{p\geq 1}$ going to infinity such that

$$\text{for all } p \geq 1, \ \ \#\mathcal{S}^\nu_{j_p}(H_\tau, \eta H_\tau) \geq b^{j_p q_\tau(\nu) H_\tau(1-\varepsilon/2)}. \tag{11}$$

Fix now $\eta \in \big(0, \min\big(\eta_0, \widetilde{C}_2\varepsilon\big)\big)$ and write $\eta = C(\varepsilon)\varepsilon$.
Let $N_0 > 0$ such that $q(n) \le q_\tau + \varepsilon^2$ for all $n \ge N_0$, and P_1 an integer such that $j_p H_\tau(1 + C_2\varepsilon) \ge N_0$ for $p \ge P_1$.
It is easily seen that there is $M > 0$ independent of ε and η such that

$$\forall\ p \ge P_1, \quad \sum_{n > j_p H_\tau(\nu)(1+C_2\varepsilon)} \ \sum_{k\in K_n} m_k \le M b^{-j_p H_\tau(1+C_2\varepsilon)(1-q_\tau-\varepsilon^2)}. \qquad (12)$$

Let $C_3 \in (\widetilde{C}(\varepsilon), \widetilde{C}_2)$ and define

$$R_p(C_3) = \Big\{\ Q \in \mathcal{S}^\nu_{j_p}(H_\tau, \eta H_\tau) : \sum_{n > j_p H_\tau(1+C_2\varepsilon)} \ \sum_{k\in K_n:\ x_k \in Q} m_k \ge b^{-j_p H_\tau(1+C_3\varepsilon)}\Big\}.$$

Also define $B_p = \mathcal{S}^\nu_{j_p}(H_\tau, \eta H_\tau) \setminus R_p(C_3)$. By construction, each element Q of B_p must contain a point x_k such that $b^{-j_p H_\tau(1+C_2\varepsilon)} \le m_k \le b^{-j_p H_\tau(1-C(\varepsilon)\varepsilon)}$. Moreover, due to (12) one has

$$\forall\ p \ge P_1,\ \#R_p(C_3) \le M b^{j H_\tau q_\tau \left[1+[C_2+(C_3-C_2)/q_\tau]\varepsilon + O(\varepsilon^2)\right]}.$$

It follows from our choice for C_2, η and C_3 that if $\varepsilon > 0$ is small enough, then

$$\forall\ p \ge P_1,\ \#R_p(C_3) \le M b^{j H_\tau q_\tau (1-\varepsilon)}.$$

This yields $\#B_p \ge b^{j_p q_\tau H_\tau (1-\varepsilon)}$ for p large enough because of (11). ∎

3.3 Theorem 1.3(1-2): Characterization of $H_\tau(\nu)$ when $q_\tau(\nu) < 1$ and proof of the linear shape of the large deviations spectrum.

Proposition 3.3 yields $q_g(\nu) = q_\tau(\nu) = q_\tau$. Let h_0 be as in Proposition 3.3. A straightforward consequence of Proposition 3.3 is that for all $\varepsilon > 0$, $\underline{f}_\nu(h_0 + \varepsilon) \ge q_\tau h_0$. Using Proposition 2.6(2), one gets $f_\nu(h_0) = q_\tau h_0$ and then $h_0 \le H_\tau(\nu)$ by Theorem 2.1(4).

Let us show first that $f_\nu(h) = q_\tau h$ for every $h \in [0, H_g(\nu)]$, and then that $H_g(\nu) = H_\tau(\nu)$ when $q_\tau < 1$.

• Let $h \in (0, h_0]$. Consider three sequences $(p_j)_{j\ge1}$, $(\varepsilon_j)_{j\ge1}$ and $(B_j)_{j\ge1}$ as in Proposition 3.3. Let $m_j = [p_j h_0/h]$. For every $Q \in B_j$, there exists a unique b-adic box Q' of generation m_j containing x_Q. Let $\varepsilon \in (0, h)$. By construction, one has $\nu(Q') \ge m_Q \ge b^{-m_j(h+\varepsilon)}$ for j large enough, and $\underline{N}(h+\varepsilon, m_j) \ge \#B_j$. Moreover, $\lim_{j\to\infty} \log_b \#B_j/m_j = q_\tau h$. So, for all $\eta > 0$ if j is large enough, $\underline{N}(h+\varepsilon, m_j) \ge b^{m_j(q_\tau h - \eta)}$. This implies, by letting $j \to +\infty$, and then ε and η go to zero, that $\underline{f}_\nu(h) \ge q_\tau h$. By Theorem 2.1(2), this yields $f_\nu(h) = q_\tau h$.

• It remains to show that $H_g(\nu) = H_\tau(\nu)$ when $q_\tau \in (0,1)$. We adopt the notations of Proposition 3.3. Let $(\varepsilon_j)_{j\geq 1}$ be a sequence of positive numbers going to 0 and such that $\sup_{Q\in B_j}\left|\frac{\log_b m_Q}{-p_j} - H_\tau(\nu)\right| \leq \varepsilon_j$ for all j. Let $(\alpha_j)_{j\geq 1}$ be a the sequence of positive numbers going to 0 such that for j large enough

$$\#B_j \geq b\, b^{p_j(H_\tau(\nu)+\varepsilon_j)} b^{-p_j(H_\tau(\nu)-\varepsilon_j)(1-q_\tau-\alpha_j)}$$

Such a choice is possible thanks to Proposition 3.3(3). Now suppose that $\forall n \in [p_j(H_\tau(\nu)-\varepsilon_j), p_j(H_\tau(\nu)+\varepsilon_j)]$, $\#K_n \cap B_j < b^{n(q_\tau-\alpha_j)}$. This yields

$$\begin{aligned}\sum_{Q\in B_j} m_Q &= \sum_{n\in[p_j(H_\tau(\nu)-\varepsilon_j),p_j(H_\tau(\nu)+\varepsilon_j)]} \sum_{Q\in K_n\cap B_j} m_Q \\ &\leq \sum_{n\in[p_j(H_\tau(\nu)-\varepsilon_j),p_j(H_\tau(\nu)+\varepsilon_j)]} b^{-(n-1)}\#(K_n\cap B_j) \\ &< \quad b.b^{-p_j(H_\tau-\varepsilon_j)\left(1-q_\tau(\nu)-\alpha_j\right)}.\end{aligned}$$

On the other hand, $\sum_{Q\in B_j} m_Q \geq b^{-p_j(H_\tau(\nu)+\varepsilon_j)}\#B_j$. Due to our choice for η_j, there is a contradiction. Hence there is an integer $n \in [p_j(H_\tau(\nu)-\varepsilon_j), p_j(H_\tau(\nu)+\varepsilon_j)]$ such that $\#K_n \cap B_j \geq b^{n(q_\tau-\alpha_j)}$. Moreover, by construction, $\mathcal{J}(n,\alpha_n) \leq p_j \leq n/(H_\tau(\nu)-\varepsilon_j)$. This implies $H_g(\nu) \geq H_\tau(\nu)$. We saw that $H_g(\nu) \leq H_\tau(\nu)$ (since $f_\nu(H_g(\nu) = q_\tau(\nu)H_g(\nu))$, hence the equality is true.

4 Theorem 1.3(3): sharpness of $H_g(\nu)$ when $q_\tau(\nu) = 1$

If $q_\tau(\nu) = q_g(\nu) = 1$ and **(H)** holds, by Theorem 2.1 one always has $f_\nu(H_\tau(\nu)) = H_\tau(\nu)$ and by the work achieved in the previous section, one also has $f_\nu(h) = h$ for every $h \in [0, H_g(\nu)]$. Hence it is natural to ask whether the large deviations spectrum is still linear for $H_g(\nu) < h < H_\tau(\nu)$. The answer is negative (as stated by item (3) of Theorem 1.3). The optimality of $H_g(\nu)$ in item (2) of Theorem 1.3 is a consequence of the examples whose constructions are detailed below. These examples depend on Propositions 4.1 and 4.2 whose long and technical proofs are available in [7]. However, to give the reader a flavor of the proof, we propose in Section 5 a one-dimensional measure for which $1/3 = H_g(\nu) < H_\tau(\nu) = 1/2$ and $f_\nu(h) = h$ for all $h \in [0, H_\tau(\nu)]$.

4.1 General construction's scheme for Theorem 1.3(3a)

Let $0 < h_0 < h_1 \leq d$. In the sequel, if $\rho \in (0,1/2]$, μ_ρ stands for the measure on $[0,1]^d$ obtained as the tensor product of d binomial measures of parameter

ρ on $[0,1]$. Recall that

$$\tau_{\mu_\rho}(q) = -d\log_2(\rho^q + (1-\rho)^q).$$

Let us consider two parameters $\rho_0 \le \rho_1$ in $(0,1/2]$, as well as μ_{ρ_0} and μ_{ρ_1} the tensor products of d binomial measures on $[0,1]$ of parameters ρ_0 and ρ_1 respectively. The parameters ρ_0 and ρ_1 can be chosen so that $\tau'_{\mu_{\rho_0}}(1) = h_0$ and $\tau'_{\mu_{\rho_1}}(1) = h_1$ (recall that $\tau'_{\mu_\rho}(1) = -d(\rho\log_2\rho + (1-\rho)\log_2(1-\rho))$.
Now, let $(\varepsilon_j)_{j\ge1}$ be a positive sequence going to 0 at ∞. For $i\in\{0,1\}$, let

$$\text{for any } j\ge 1,\ \ E^i_j = \bigcap_{j'\ge j} \ \bigcup_{Q\in\mathcal{G}_{j'}:\ 2^{-j'h_i(1+\varepsilon_{j'})}\le \mu_{\rho_i}(Q)\le 2^{-j'h_i(1-\varepsilon_{j'})}} Q.$$

It is well-known that μ_{ρ_i} is carried by the set $\widehat{E}^{\rho_i}_{h_i}$ (see Remark 2.3 for the definition of this set). Thus the sequence $(\varepsilon_j)_{j\ge1}$ can be chosen so that $\mu_{\rho_i}\left(\bigcup_{j\ge1} E^i_j\right) = 1$ for $i\in\{0,1\}$. Also the sets E^i_j form a non-decreasing sequence. One fixes l_i, $i\in\{0,1\}$ such that $\mu_{\rho_i}(E^i_{l_i}) \ge 1/2$. Let us consider, for $j\ge1$, the subset $\mathcal{G}^{(i)}_j$ of intervals of $\mathcal{G}_j$ defined by $\mathcal{G}^{(i)}_j = \{Q\in\mathcal{G}_j : Q\cap E^i_{l_i} \ne \emptyset\}$. Notice that by construction one has $\lim_{j\to\infty} \frac{\log_2 \#\mathcal{G}^{(i)}_j}{j} = h_i$ for $i\in\{0,1\}$.

For $n\ge1$, we build the sequence of purely discontinuous measures

$$\nu^0_n = \sum_{Q\in\mathcal{G}^{(0)}_n} \mu_{\rho_0}(Q)\delta_{x_Q}.$$

Set $j_1=2$, $n_1=4$, and for every $k\ge2$, $j_k = 2^{2^{n_{k-1}}}$ and then $n_k = 2^{j_k}$. When k is large, $n_{k-1} = o(j_k)$ and $j_k = o(n_k)$. Then define

$$\nu = \sum_{k\ge1} 2^{-k} \sum_{Q\in\mathcal{G}^{(1)}_{j_k}} \mu_{\rho_1}(Q)\,\nu^0_{n_k}\circ f_Q^{-1},$$

where f_Q stands for a similitude mapping $[0,1]^d$ onto Q. In particular, notice that the Dirac masses used in this construction take values $2^{-k}\mu_{\rho_1}(Q_k)\mu_{\rho_0}(Q'_k)$ at $x_{f_{Q_k}(Q'_k)}$, with $(Q_k,Q'_k)\in\mathcal{G}^{(1)}_{j_k}\times\mathcal{G}^{(0)}_{n_k}$. Then

Proposition 4.1 *One has $q_\tau(\nu) = q_g(\nu) = 1$, $H_\tau(\nu) = h_1$, $H_g(\nu) = h_0$, and $f_\nu(h) = h$ for every $h\in[0,h_1]$.*

4.2 General construction's scheme for Theorem 1.3(3b)

We adopt the notations of the previous section and suppose that $h_0 < h_1$. Let $(\theta_k)_{k\ge1}$ be an increasing sequence of integers such that $\theta_k j_k = o(n_k)$ as

$k \to \infty$. Then let

$$\text{for } k \geq 1, \quad \mu^{\theta_k} = \sum_{Q \in \mathcal{G}_{\theta_k j_k}} \mu_{\rho_1}(Q)\, \mu_{p_0} \circ f_Q^{-1}.$$

Now for $k, n \geq 1$ consider the measure $\nu_n^{\theta_k} = \sum_{Q \in \mathcal{G}_n^{(0)}} \mu^{\theta_k}(Q) \delta_{x_Q}$. Finally let

$$\nu = \sum_{k \geq 1} 2^{-k} \sum_{Q \in \mathcal{G}_{j_k}^{(1)}} \mu_{\rho_1}(Q)\, \nu_{n_k}^{\theta_k} \circ f_Q^{-1}.$$

Proposition 4.2 *One has $q_\tau(\nu) = q_g(\nu) = 1$, $H_\tau(\nu) = h_1$, $H_g(\nu) = h_0$, and $f_\nu(h) < h$ for every $h \in (h_0, h_1)$.*

4.3 Theorem 1.3(4): A condition to have $H_g(\nu) = H_\tau(\nu)$

Let $\varepsilon \in (0, H_\tau(\nu)^{-1})$. Let (n_j) as in the statement. Let $\eta > 0$ and suppose that there exists an integer j_0 such that for $j \geq j_0$ the set X_{n_j} is included in the union of less than $b^{n_j(1-\eta)}$ b-adic boxes of generation $[n_j(H_\tau(\nu)^{-1} - \varepsilon)]$. This implies that $\bigcup_{k \in K_{n_j}} B(x_k, b^{-n_j(H_\tau(\nu)^{-1} - \varepsilon)})$ is covered by at most $3^d b^{n_j(1-\eta)}$ b-adic boxes of generation $[n_j(H_\tau(\nu)^{-1} - \varepsilon)]$. Elementary computations yield

$$\dim \left(\limsup_{j \to \infty} \bigcup_{k \in K_{n_j}} B(x_k, b^{-n_j(H_\tau(\nu)^{-1} - \varepsilon)}) \right) \leq \frac{1-\eta}{H_\tau(\nu)^{-1} - \varepsilon}.$$

If $\eta \in (H_\tau(\nu)\varepsilon, 1)$, then $\frac{1-\eta}{H_\tau(\nu)^{-1} - \varepsilon}$. This yields a contradiction with our assumption. Consequently, there is an increasing sequence $(j_p)_{p \geq 1}$ such that $\forall p \geq 1$ the set $X_{n_{j_p}}$ is included in the union of at least $b^{n_{j_p}(1-\eta)}$ b-adic boxes of generation $[n_{j_p}(H_\tau(\nu)^{-1} - \varepsilon)]$. We let the reader check this implies $H_g(\nu) \geq H_\tau(\nu)$.

5 A simple example illustrating Theorem 1.3(3a)

The idea is to replace the binomial measures in Section 4 by uniform measures on Cantor sets, which are known to be monofractal and are easier to deal with.

(i) Preliminary step: Let us consider the Cantor set K_0 defined by the recursive scheme: Consider the interval $I = [0, 1]$. Divide it into 8 intervals of length $1/8$, and keep only the first and last ones, denoted respectively by I_0 and I_1. The same dividing scheme applied to I_0 (*resp.* I_1) yields two intervals $I_{0,0}$ and $I_{0,1}$ (*resp.* $I_{1,0}$ and $I_{1,1}$) of length $(1/8)^2$. Iterating this procedure yields, at every generation $m \geq 1$, 2^m intervals of same length $(1/8)^n = (1/2)^{3n}$. Let us denote $E^0_m \subset \mathcal{G}_{3m}$ this set of intervals, and $K_0 = \bigcap_{m\geq 1} E^0_m$.

For every $m \geq 1$, consider the probability measure $\mu^0(m)$ which is uniformly distributed on E^0_m, i.e. $\mu^0(m)$ has a density $f_{\mu^0(m)}$ equal to

$$f_{\mu^0(m)} = \sum_{I\in E^0_m} 2^{-m}\mathbf{1}_I(x), \text{ where } \mathbf{1}_I \text{ is the indicator function of the interval } I,$$

and the discontinuous measure $\nu^0(m)$ defined by

$$\nu^0(m) = \sum_{I\in E^0_m} 2^{-m}\delta_{x_I}, \text{ where } x_I \text{ is the left end-point of the interval } I. \quad (13)$$

Similarly, consider the Cantor set K_1 where each interval is split into only four equal parts and where the two extremal intervals are kept at each generation. Again, the m-th generation of the construction is denoted $E^1_m \subset \mathcal{G}_{2m}$ and $K_1 = \bigcap_{m\geq 1} E^1_m$. Finally, for every $m \geq 1$ two measures $\mu^1(m)$ and $\nu^1(m)$ are built using the same scheme as the one used for $\mu^0(m)$ and $\nu^0(m)$.

The reader can verify next lemma, which follows from classical self-similarity properties of the construction and of the uniform measure on Cantor sets.

Lemma 5.1 *Set $h_0 = 1/3$ and $h_1 = 1/2$. For every $m \geq 2$,*

1. *If I is a dyadic interval of generation $1 \leq j \leq 3m$, then either $\nu^0(m)(I) = \mu^0(m)(I) = 0$ or $|I|^{h_0}/8 \leq \nu^0(m)(I) = \mu^0(m)(I) \leq 8|I|^{h_0}$. The cardinality N^0_j of the set $\{I \in \mathcal{G}_j : \nu^0(m)(I) > 0\}$ satisfies $2^{jh_0}/8 \leq N^0_j \leq 8.2^{jh_0}$.*

2. *If I is a dyadic interval of generation $1 \leq j \leq 2m$, then either $\nu^1(m)(I) = \mu^1(m)(I) = 0$ or $|I|^{h_1}/4 \leq \nu^1(m)(I) = \mu^1(m)(I) \leq 4|I|^{h_1}$. The cardinality N^1_j of the set $\{I \in \mathcal{G}_j : \nu^1(m)(I) > 0\}$ satisfies $2^{jh_1}/4 \leq N^1_j \leq 4.2^{jh_1}$.*

3. *For any $\varepsilon > 0$, there exists an integer m_ε such that for every $m \geq m_\varepsilon$, any subset $E \subset \{x_I : I \in E^0_m\}$ of cardinality greater than $2^{m(1-\varepsilon/2)} = (\#E^0_m)^{1-\varepsilon}$ contains two points x and y such that $|x-y| \leq 2^{-3m(1-\varepsilon)}$.*

(ii) Construction of the measure ν: Two sequences of integers $(j_k)_{k\in\mathbb{N}}$ and $(n_k)_{k\in\mathbb{N}}$ are needed. They are built recursively, using the same scheme as in Section 4: Set $j_1 = 2$, $n_1 = 4$, and $\forall k \geq 2$ $j_k = 2^{2^{n_{k-1}}}$ and then $n_k = 2^{j_k}$.

For every dyadic interval I, we denote by f_I the increasing affine mapping which maps I onto $[0,1]$. Finally one sets

$$\nu = \sum_{k\geq 1} \frac{1}{j_k} \sum_{I\in E^1_{j_k}} 2^{-j_k}\nu^0(n_k)\circ f_I. \quad (14)$$

Remark that:

• The Dirac masses which appear in (14) all have an intensity of order $2^{-(j_k+n_k)}/j_k$ and when k is fixed, there are $2^{j_k+n_k}$ such Dirac masses. Moreover, still for a given $k\geq 1$, these masses are located at dyadic points of scale $2^{-(2j_k+3n_k)}$.

• For every $k\geq 1$, the Dirac masses of generation $k'\geq k$ which appear in the sum (14) all belong to one of the intervals $I\in E^1_{j_k}$.

• For every $k\geq 1$, one has (recall that $j_{k+1}=2^{2^{n_k}}$, $n_{k-1}=o(j_k)$ and $j_k=o(n_k)$)

$$\sum_{k'\geq k+1}\frac{1}{j_k}\sum_{I\in E^1_{j_k}} 2^{-j_k}=o(2^{-(j_k+n_k)}). \tag{15}$$

• The structure of ν is comparable to the one of Cantor sets with different upper and lower box dimensions (see [12]).

(iii) Properties of ν: They follow from Propositions 5.2 and 5.3.

Proposition 5.2 *One has $q_g(\nu)=q_\tau(\nu)=1$, $H_\tau(\nu)=h_1$ and $H_g(\nu)=h_0$.*

Proof: (i) At every scale $j=2j_k$, there are by construction 2^{j_k} intervals whose ν-mass is larger than $2^{j_k}/j_k$. This holds for every $k\geq 1$, and thus $\underline{f}_\nu(h_1)\geq h_1$. By Proposition 2.6, $q_\tau(\nu)=1$ and $f_\nu(h_1)=h_1$, and finally $H_\tau(\nu)\geq h_1$.

In order to get $H_\tau(\nu)=h_1$, it is enough to prove that $\tau_\nu(q)\geq (q-1)h_1$ near 1^- (indeed, this clearly implies that $H_\tau(\nu)=\tau'_\nu(1^-)\leq h_1$).
Let $q\in(0,1)$ and $j\geq 1$. Let k be the unique integer such that $2j_k\leq j<2j_{k+1}$.
- If $2j_k\leq j<2j_k+3n_k$**:** Let us evaluate $s_j(q)$ (defined in (2)) for the measure ν. There are only types of dyadic intervals of non-zero ν-mass at scale j:

- Those which contain a Dirac mass of the form $2^{-(j_{k'}+n_{k'})}/j_{k'}$ for $k'\in\{1,...,k-1\}$. For each such k', there are at most $2^{-(j_{k'}+n_{k'})}$ of them.

- By item (1) of Lemma 5.1, if an interval of scale j does not contain a mass of generation $<k$, then either its ν-mass is 0, or it is equivalent to $2^{-(j-2j_k)(h_0)}2^{-j_k}/j_k$ (one implicitly uses (15) which ensures that the masses of next generations do not interfere). Still by Lemma 5.1, the number of such intervals is $N^0_{j-2j_k}$, which is approximately $2^{(j-2j_k)h_0}$.

Due to the sub-additivity of the function $x\mapsto x^q$ on $\mathbb{R}_+$ ($q<1$), one has

$$\begin{aligned} s_j(q)=\sum_{I\in\mathcal{G}_j}\nu(I)^q &\leq \sum_{k'=1}^{k-1} 2^{j_{k'}+n_{k'}}\Big(\frac{2^{-(j_{k'}+n_{k'})}}{j_{k'}}\Big)^q+N^0_{j-2j_k}\Big(\frac{2^{-(j-2j_k)(h_0)}.2^{-j_k}}{j_k}\Big)^q\\ &\leq C\Big(2^{(j_{k-1}+n_{k-1})(1-q)}+2^{(j-2j_k)(1-q)-qj_k}\Big). \end{aligned}$$

Given $\varepsilon > 0$, using that $j_{k-1} + n_{k-1} = o(j_k)$, taking the log and then dividing by $-j$, one finds that as soon as j (and thus k) is large enough,

$$\frac{\log s_j(q)}{-j} \geq h_0(q-1) + q\frac{2j_k}{j} - \varepsilon.$$

which is always greater than $h_0(q-1) - \varepsilon$, for every ε when k is large enough.

- If $2j_k + 3n_k \leq j < 2j_{k+1}$**:** Again there are two types of intervals:

- Those which contain a Dirac mass of the form $2^{-(j_{k'}+n_{k'})}/j_{k'}$ for $k' \in \{1, ..., k\}$. For each $k' \in \{1, ..., k\}$, there are at most $2^{-(j_{k'}+n_{k'})}$ of them.
- By item (2) of Lemma 5.1, if an interval of scale j does not contain a mass of generation $\leq k$, then its ν-mass is either 0 or is equivalent to $|I|^{h_1}/j_{k+1} = 2^{-jh_1}/j_{k+1}$ (again, (15) is used). Still by Lemma 5.1, the number of such intervals is N_j^1.

Hence the same estimates as above yields

$$s_j(q) \leq \sum_{k'=1}^{k} 2^{j_{k'}+n_{k'}} \Big(\frac{2^{-(j_{k'}+n_{k'})}}{j_{k'}}\Big)^q + N_j^1 \Big(\frac{2^{-jh_1}}{j_{k+1}}\Big)^q \leq C\Big(2^{(j_k+n_k)(1-q)} + \frac{2^{jh_1(1-q)}}{j_{k+1}^q}\Big).$$

Given $\varepsilon > 0$, using that $j_{k-1} + n_{k-1} = o(j_k)$, taking the log and then dividing by $-j$, one finds that if j (and thus k) is large enough,

$$\frac{\log s_j(q)}{-j} \geq (q-1)\Big(h_1 + \frac{j_k + n_k}{j}\Big) + q\frac{\log j_{k+1}}{j} \geq (q-1)\Big(h_1 + \frac{j_k + 2^{j_k}}{j}\Big) + q\frac{2^{n_k}}{j}.$$

Let $\varepsilon > 0$. Since $j_k = o(n_k)$, when k is large enough, one has

$$\frac{\log s_j(q)}{-j} \quad \geq \quad h_1(q-1) + q\frac{2^{n_k(1-\varepsilon)}}{j}.$$

which is always greater than $h_1(q-1)$ (and actually which is equivalent to $h_1(q-1)$ when j is close to $2j_{k+1}$).

(ii) To finish with the proof we need to establish that $H_g(\nu) = h_0$.

For every $k \geq 1$, let us denote by l_k the unique integer such that $2^{-l_k} \leq 2^{-(j_k+n_k)}/j_k \leq 2^{-l_k+1}$. By construction, for any integer $n \geq 1$, either $q(n) = 0$ or there is k such that $n = l_k$ and thus $q(n) = q(l_k) = \frac{\log_2 2^{j_k+n_k}}{l_k}$, which clearly tends to 1 when k goes to infinity. Hence $q_g(\nu) = q_\tau(\nu) = 1$.

Let $\widetilde{\alpha} = (\alpha_k)_{k\in\mathbb{N}}$ be a positive sequence converging to 0. Let K be such that for every $k \geq K$, $\alpha_k \leq \varepsilon/4$.

Consider such an integer $k \geq K$. Let E be any subset of $X(l_k)$ of cardinality greater than $(\#X(l_k))^{(1-\alpha_k)} = 2^{(j_k+n_k)(1-\alpha_k)}$. By construction (self-similarity of the Cantor set), it is obvious that $\mathcal{J}(l_k, \alpha_k) \geq 2j_k + 3n_k$

The points of $X(l_k)$ can be separated into 2^{j_k} packets of 2^{n_k} Dirac masses, where each packet corresponds to one term $\nu^0(n_k) \circ f_I$ ($I \in E^1_{j_k}$) in the definition of ν (14). As a consequence, there is one packet such that the set E contains (at least) $2^{(j_k+n_k)(1-\alpha_k)}/2^{j_k}$ of the initial Dirac masses of this packet. Since $j_k = o(n_k)$, for k large enough, E contains at least $2^{n_k(1-\varepsilon/2)}$ Dirac masses.

By item (3) of Lemma 5.1, any such subset $E \subset X(l_k)$ contains two points x and y such that $|x-y| \leq 2^{-j_k} 2^{-3n_k(1-\varepsilon)}$. Hence $\mathcal{J}(l_k, \alpha_k) \leq j_k + 3n_k(1-\varepsilon)$.

Using the two bounds for $\mathcal{J}(l_k, \alpha_k)$, one gets that $\lim_{k\to+\infty} \frac{l_k}{\mathcal{J}(l_k,\alpha_k)} = 1/3 = h_0$. Hence $H_g(\nu) = h_0$. This ends the proof. ■

Proposition 5.3 *For every $h \in (h_0, h_1)$ one has $f_\nu(h) = h$.*

Proof: Let $k \geq 1$ be large enough and $j \geq 1$ be such that $2j_k \leq j < 2j_k + 3n_k$. Let $\varepsilon > 0$. As explained above in Proposition 5.2, there are, at scale j, at least $2^{(j-2j_k)h_0(1-\varepsilon)} 2^{-j_k}$ intervals I of length 2^{-j} such that $\nu(I) \geq 2^{-(j-2j_k)h_0} 2^{-j_k}/j_k$. But $2^{-(j-2j_k)h_0} 2^{-j_k}/j_k = 2^{-jh_j}$, where $h_j^{(k)} = h_0 + j_k/j(1-2h_0) + (\log_2 j_k)/j = h_0 + j_k/j(1-2h_0) + n_{k-1}/j$. Remark that the exponents $h_j^{(k)}$ range in $[h_0, h_0 + 1/2 - h_0] = [h_0, h_1]$ when j describes $\{2j_k, ..., 2j_k + 3n_k\}$ and that any $h \in [h_0, h_1]$ is the limit of a sequence of such points $h_j^{(k)}$ fully when $k \to +\infty$.

Let $h > 0$ and assume that $\varepsilon' > 0$ is small enough so that $[h-\varepsilon', h+\varepsilon'] \subset (h_0, h_1)$. Assume also that k is large enough so that there is $j \in [2j_k + 3n_k, 2j_{k+1}]$ such that $h_j^{(k)} \in [h-\varepsilon', h+\varepsilon']$. As proved just above, the number of intervals I of scale j such that $\nu(I) \geq 2^{-jh_j}$ is greater than $2^{(j-2j_k)h_0(1-\varepsilon)} 2^{j_k} \geq 2^{jh_j(1-\varepsilon)}$. This occurs for an infinite number of scales j and for every $\varepsilon' > 0$ and $\varepsilon > 0$, hence $\underline{f}_\nu(h) \geq h$. By item (2) of Proposition 2.6, $f_\nu(h) = h$. ■

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