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From multifractal measures to multifractal wavelet series

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ABSTRACT. *Given a positive locally finite Borel measure μ on $\mathbb{R}$, a natural way to construct multifractal wavelet series $F_\mu(x) = \sum_{j\geq 0, k\in\mathbb{Z}} d_{j,k}\psi_{j,k}(x)$ is to set $|d_{j,k}| = 2^{-j(s_0-1/p_0)}\mu([k2^{-j},(k+1)2^{-j}))^{1/p_0}$, where $s_0, p_0 \geq 0$, $s_0 - 1/p_0 > 0$. Indeed, under suitable conditions, it is shown that the function F_μ inherits the multifractal properties of μ.*

The transposition of multifractal properties works with many classes of statistically self-similar multifractal measures, enlarging the class of processes which have self-similarity properties and controlled multifractal behaviors.

Several perturbations of the wavelet coefficients and their impact on the multifractal nature of F_μ are studied. As an application, multifractal Gaussian processes associated with F_μ are created. We obtain results for the multifractal spectrum of the so-called $\mathcal{W}$-cascades introduced by Arnéodo et al.

1. Introduction and motivations

Phenomena exhibiting wild regularity variations are now well identified in many areas. For instance, they occur in fluid mechanics (intermittent turbulence [40, 24]), in traffic analysis (road and Internet traffic [39]), and in finance [42]. Modeling these phenomena is a major issue for further applications. In particular, finding processes with a local regularity that can be controlled is an active domain of research. Among these processes, those having properties of statistical self-similarity and of stability after perturbations are of special interest. They are easier to study, since many works have already investigated the subject. They also are better candidates to

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fit data from areas listed above, where scaling invariances play important roles.

When they fulfill these conditions, most of the time these processes satisfy some *multifractal formalism*, either for functions [47, 27] or for measures [16, 48, 21, 10]. Multifractal formalisms take their origin in the study of fluid mechanics and dynamical systems [24, 25, 17], and are closely related to thermodynamical formalism. Before introducing this concept, let us explain how the local behavior of a continuous function is measured in this paper. Given a non-trivial open interval I of $\mathbb{R}$, the local regularity at a point $x_0 \in I$ of a function $f \in L^\infty_{loc}(I)$ is given by the pointwise Hölder exponent $h_f(x_0)$, defined as follows. The function f belongs to $C^h_{x_0}$ if and only if there exist a constant C and a polynomial P of degree smaller than $[h]$ such that

$$\forall\, x \in I \text{ close enough to } x_0,\ \ |f(x) - P(x-x_0)| \leq C|x-x_0|^h.$$

The *pointwise Hölder exponent* of f at x_0 is

$$h_f(x_0) = \sup\{h : f \in C^h_{x_0}\}. \tag{1.1}$$

Multifractal analysis then focuses on the dimension of the (often fractal) level sets of the function h_f, that is the sets of the form

$$E^f_h = \{x \in I : h_f(x) = h\} \quad (h \geq 0). \tag{1.2}$$

The most common notion of dimension is the Hausdorff dimension, denoted dim in this paper. The Hausdorff multifractal spectrum of f is defined by

$$d_f : h \mapsto \dim E^f_h. \tag{1.3}$$

The same notions of level sets and spectrum are associated with any positive Borel measure μ on $\mathbb{R}$, for which the local regularity at a given point x is given by another Hölder exponent $h_\mu(x)$ defined by

$$h_\mu(x) = \liminf_{r\to 0^+} \frac{\log \mu(B(x,r))}{\log r}.$$

The knowledge of these multifractal spectra yields a geometrical idea of the repartition of the singularities of the initial function or measure. Unfortunately, this theoretical point of view is not adapted to numerical simulations or data processing, since Hausdorff dimensions are not reachable numerically.

A multifractal formalism is a heuristic formula relating, via a Legendre transform, the Hausdorff multifractal spectrum of a function f (or a measure) to some kind of free energy function (or "scaling function") η_f

associated with f. When this Legendre transform precisely yields the Hausdorff spectrum, the multifractal formalism is said to hold for f. The main interest of this formalism for physicists is the following: If a given signal f is supposed to fulfill the multifractal formalism, then its multifractal spectrum, which contains deep local information, can be approximated by using an estimation of the scaling function η_f. The crucial point is that the scaling function is numerically accessible. As we already said, the validity of multifractal formalisms has been established for wide classes of measures and functions possessing statistical self-similarity properties.

There are several ways to define the scaling function associated with a continuous function, most of them using wavelet decompositions [47, 27, 32]. Let ψ be a function in the Schwartz class, as constructed in [34] or [44]. The set of functions $\{\psi_{j,k} = \psi(2^j \cdot -k)\}$, where $(j,k) \in \mathbb{Z}^2$, forms an orthogonal basis of $L^2(\mathbb{R})$, and any function $f \in L^2(\mathbb{R})$ can be written

$$f(x) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} d_{j,k}\psi_{j,k}(x),$$

where $d_{j,k}$ is the wavelet coefficient of f defined by

$$d_{j,k} := d_{j,k}(f) = 2^j \int_{\mathbb{R}} f(t)\psi_{j,k}(t)dt.$$

Wavelets are natural tools in multifractal analysis, for at least three reasons. First, the concept of self-similarity is implicit in the construction of the wavelet basis $\{\psi_{j,k}\}_{j,k}$. Second, wavelet coefficients provide a time-scale decomposition of the initial function (or signal) f. Hence scaling properties of a function shall imply scaling properties of its wavelet coefficients. Finally, the pointwise Hölder exponent $h_f(x)$ of any continuous function f around a point x can be computed through size estimates of the wavelet coefficients $d_{j,k}$ associated with f (see Section 2). Thus they are efficient tools to analyze local behaviors.

Wavelet also provide an appropriate frame to generate processes with scaling properties and which multifractal structure can be controlled [3, 27, 29, 30, 4]. In this article, we propose a natural construction of continuous functions F_μ based on a measure μ and on a wavelet basis $\{\psi_{j,k}\}$. Namely, given a positive Borel measure μ on $\mathbb{R}$, the function F_μ is defined as a wavelet series given by

$$F_\mu(x) = \sum_{j\geq 0} \sum_{k\in\mathbb{Z}} \pm 2^{-j(s_0-1/p_0)} \mu\big([k2^{-j}, (k+1)2^{-j})\big)^{1/p_0} \psi_{j,k}(x), \tag{1.4}$$

where $s_0, p_0 \geq 0$, $s_0 - 1/p_0 > 0$. These two positive real parameters rule the global regularity of F_μ.

This wavelet series model, and especially some of the generalizations we propose in the rest of the paper, are implicitly used in several situations, especially in fluids mechanics and study of fully developed turbulence (see for instance [3, 15]), as well as in traffic analysis [50]. In the following sections, these applications appear as simple perturbations of our wavelet series F_μ when the measure μ used for the construction is an independent random cascade. Our approach enables us to compute the Hausdorff multifractal spectra of the wavelet series and their perturbations in several cases, while this was not realized in the works mentioned above.

We prove that the control of the multifractal structure of μ yields a control on the multifractal structure of F_μ. We shall use a slight modification of the multifractal formalism for measures of [16], as well as the multifractal formalism for functions of [32]. Our main result is

Theorem 1. *Let μ be a positive Borel measure on $[0,1]$. Let $s_0, p_0 \geq 0$, $s_0 - 1/p_0 > 0$, and consider the wavelet series F_μ (1.4) associated with μ.*

If μ obeys the multifractal formalism for measures at singularity $\alpha \geq 0$, then F_μ obeys the multifractal formalism for functions at $h = s_0 - 1/p_0 + \alpha/p_0$, and $d_{F_\mu}(h) = d_\mu(\alpha)$.

This result provides us with a simple and satisfactory bridge between multifractal analysis of measures and multifractal analysis of functions.

Let us emphasize that, although built on a dyadic grid, the multifractal formalism for measures we use is satisfied by measures with a construction not based on the dyadic grid. In particular, Theorem 1 can be applied to the classical families of multifractal measures μ generated by multiplicative procedures, like for example quasi-Bernoulli measures [16] and Mandelbrot b-adic random multiplicative cascades [40]. It also applies to compound Poisson cascade measures [9] and their extensions [5, 10] as well as to stable Lévy measures [14, 28]. When μ is random, we exhibit cases where almost surely the whole multifractal spectrum of F_μ can be computed (and not only each point of this spectrum almost surely). In each case, the verification needs non-obvious arguments. We detail hereafter the cases of dyadic random multiplicative cascades and of stable Lévy measures. The properties required to deal with the examples that we mentioned above, but that we do not treat in this paper, can be found in [12, 13].

An important property of the construction is its stability after perturbations of the wavelet coefficients. Indeed, it is shown that, under reasonable assumptions, a part of the multifractal spectrum remains unchanged. This gives rise to important applications. For example, the famous $\mathcal{W}$-cascades of Arnéodo *et al* in [3] can now be seen as a perturbation of a wavelet series F_μ associated with a well-chosen random multiplicative cascade measure μ. Using this interpretation, under suitable assumptions, we obtain almost surely the whole multifractal spectrum conjectured in [3] for this class of random wavelet series, while only their global regularity was explicitly computed.

Let us mention another application. Given a measure μ satisfying the multifractal formalism, one can explicitly construct Gaussian processes which multifractal spectra are deduced from the one of μ by affine transformations.

Perturbing the construction is also a way to simplify the simulation of multifractal functions having the same spectrum as F_μ. Indeed, a multifractal measure μ is often the limit when $j \to +\infty$ of some simple measure-valued sequence $\{\mu_j\}_{j\geq 0}$. Then a convenient perturbation is often to replace $\mu([k2^{-j},(k+1)2^{-j})$ by $\mu_j([k2^{-j},(k+1)2^{-j})$ in the construction of F_μ ($\mathcal{W}$-cascades are obtained like this).

Let us detail a last application. In [11], a class of discontinuous measures was introduced. Given an initial positive Borel measure μ, these new multifractal measures ν constituted only by Dirac masses have the form

$$\nu = \sum_{j\geq 1} j^{-2} \sum_{k=0,\dots,2^j-1} \mu([k2^{-j},(k+1)2^{-j}))\,\delta_{k2^{-j}}. \tag{1.5}$$

Among other properties, these measures fulfill the multifractal formalism we use in this paper (see Remark 2), and Theorem 1 can be applied to the wavelet series F_ν. The reader can verify that when μ is the Lebesgue measure, the corresponding wavelet series F_ν built with parameters $s_0>0$, $p_0>0$, $s_0-1/p_0>0$, are simple perturbations of the so-called saturating functions used by S. Jaffard in [30] to prove the genericity in the sense of Baire's categories of multifractal functions in Besov spaces $B^{s_0,\infty}_{p_0}(\mathbb{R})$.

Jaffard [29], Aubry and Jaffard [4], created processes with wavelet coefficients that are mutually independent and identically distributed random variables. They reach non-decreasing Hausdorff multifractal spectra, nowhere strictly concave. Moreover, these processes have oscillating singularities. When working on real data, due to the use of the Legendre transform, concave spectra with a decreasing part are often encountered. Our construction, as well as the one of [3], reaches functions with theoretical strictly concave Hausdorff spectra (see Section 6), with a non-trivial decreasing part (not only in the Legendre spectrum). This certainly comes from the strong correlations between the wavelet coefficients of F_μ, and could lead to more realistic models. These strong correlations also imply that the wavelet series F_μ has no oscillating singularity. This corroborates the fact that the validity of multifractal formalism most often implies that there are no major oscillations phenomena in the studied object [43, 52, 53].

Before proving Theorem 1, some recalls on both multifractal analysis of functions and of measures is needed. Section 2 concerns functions: It provides the multifractal formalism for functions [31, 32, 33] well adapted to our construction.

Section 3 introduces a modified version of the multifractal formalism for measures of [16]. Indeed, the single usual Hölder exponent for measures does not provide enough information to control the regularity of the functions we build. Therefore the definitions of the usual level sets E^μ_α must be modified. Sufficient conditions for this modified multifractal formalism to hold are given in Theorem 4.

The wavelet series F_μ is defined and studied in Section 4. Perturbations of the wavelet coefficients of F_μ are studied in Section 5. Section 6 provides fundamental examples of qualified measures μ and of associated functions F_μ. It also contains the application to $\mathcal{W}$-cascades. Section 7 contains the proof of Theorem 4. Eventually, Section 8 is devoted to the proofs of the results stated in Section 6.

2. Functions setting

As explained before, the decomposition of functions on orthonormal wavelet bases is fundamental in our approach. Let ψ be a function in the Schwartz class, as constructed in [34] or [44]. We mention that ψ can also be chosen with compact support, see [19]. Nevertheless such a choice introduces technical complications unuseful to our purpose. For instance, if a compactly supported wavelet ψ is used, outside the support of μ, even if the wavelet is smooth enough and has enough vanishing moments, the regularity of the series F_μ we build is governed by the one of ψ.

Thus, in the sequel, the wavelet ψ is fixed and belongs to $C^\infty(\mathbb{R})$. Moreover, all its moments of positive orders are supposed to be null.

Recall that the notions of pointwise Hölder exponent, level sets and Hausdorff multifractal spectrum of a function f have been introduced in the previous section ((1.1), (1.2),(1.3)).

Remark 1. In Sections 4-6, if a compactly supported wavelet were used (instead of a C^∞ wavelet), the results below would be valid by replacing E^f_h by $E^f_h \cap$ supp(μ), and if the wavelet $\psi \in C^{s-1/p+\alpha_{\max}/p}(\mathbb{R})$, where $\alpha_{\max}$ is the largest Hölder exponent of μ.

2.1 Pointwise Hölder exponent and wavelet leaders

We recall the definitions of wavelet leaders and the associated result of S. Jaffard in [32].

For any couple $(j,k) \in \mathbb{N}\times\mathbb{Z}$, $I_{j,k}$ denotes the dyadic interval $[k2^{-j},(k+1)2^{-j})$. Then, if $x \in \mathbb{R}$, $\forall j \geq 1$, there exists a unique integer $k_{j,x}$ such that $x \in I_{j,k_{j,x}}$. The interval $I_{j,k_{j,x}}$ is also denoted $I_j(x)$.

Let $f \in C^\varepsilon(\mathbb{R})$ for some $\varepsilon > 0$, and write it in the wavelet basis

$$f = \sum_{j,k \in \mathbb{Z}} d_{j,k} \psi_{j,k}. \tag{2.1}$$

For any couple $(j,k) \in \mathbb{N} \times \mathbb{Z}$, let us introduce the wavelet leader $L_{j,k}$ associated with f and ψ

$$L_{j,k} = \sup_{j' \geq j,\ k' 2^{-j'} \in I_{j,k}} |d_{j',k'}|. \tag{2.2}$$

Then, with any point $x_0 \in \mathbb{R}$ and any scale $j \geq 0$ can be associated the coefficient

$$L_j(x_0) = \sup_{|k - k_{j,x}| \leq 1} L_{j,k}.$$

Theorem 2. *Assume that f belongs to $C^\varepsilon(\mathbb{R})$ for some $\varepsilon > 0$. Let ψ be a function in the Schwartz class, as constructed in [34]. Then for any $x_0 \in \mathbb{R}$,*

$$h_f(x_0) = \liminf_{j \to +\infty} \frac{\log L_j(x_0)}{\log 2^{-j}}. \tag{2.3}$$

Theorem 2, proved in [32], provides us with a wavelet characterization of the pointwise Hölder exponent for a uniform Hölder function.

2.2 Upper bound for $d_f(h)$ and multifractal formalism

Recall that the Legendre transform of a function $\varphi : q \in \mathbb{R} \mapsto \varphi(q)$ is the mapping

$$\varphi^* : h \in \mathbb{R} \mapsto \varphi^*(h) = \inf_{q \in \mathbb{R}} (qh - \varphi(q)) \in \mathbb{R} \cup \{-\infty\}. \tag{2.4}$$

For any function $f \in L^2(\mathbb{R})$ decomposed into (2.1), one can introduce the scaling function ξ_f associated with f

$$\xi_f : p \in \mathbb{R} \mapsto \xi_f(p) = \liminf_{j \to +\infty} \frac{\log \sum^*_{k \in \mathbb{Z}} |L_{j,k}|^p}{\log 2^{-j}}, \tag{2.5}$$

where $*$ means that the sum is taken over the k such that $|L_{j,k}|$ does not vanish. Since for each $j \geq 1$ the function $p \mapsto \sum^*_{k \in \mathbb{Z}} |L_{j,k}|^p$ is log-convex and non-increasing when j is large enough, the mapping ξ_f is concave and non-decreasing on $\mathbb{R}$ (as a the limit of the infimum of non-decreasing concave functions).

This kind of free energy function are naturally introduced in order to formulate a multifractal formalism for functions based on the representation as wavelet series (see [47, 27, 31] for example). Frisch and Parisi first proposed in [24] a formula that links the multifractal spectrum of a function f with some averaged quantities derived from f. This formula, generically referred to as the *Frisch-Parisi conjecture*, can be generalized and reformulated in (see [24, 27, 30])

$$d_f(h) = \inf_{p>0} \left(ph - \eta_f(p)\right) = (\eta_f)^*(h), \tag{2.6}$$

where the mapping $\eta_f : p \in \mathbb{R} \mapsto \mathbb{R} \cup \{-\infty\}$ is a suitable free energy function associated with f.

In the following we take $\eta_f \equiv \xi_f$. At this stage ξ_f also depends on ψ, and of course, (2.6) does not always hold. Nevertheless, Jaffard establishes the following theorem [31, 32]

Theorem 3. *Assume that the function f belongs to $C^\varepsilon(\mathbb{R})$ for some $\varepsilon > 0$. Let ψ be a function in the Schwartz class, as constructed in [34]. Then the scaling function ξ_f depends only on f, not on ψ. Moreover, one has*

$$\text{for all } h \geq 0, \;\; d_f(h) \leq \inf_{p\in\mathbb{R}} (hp - \xi_f(p)) = (\xi_f)^*(h). \tag{2.7}$$

As a consequence, Theorem 3 yields a generic upper bound for the multifractal spectrum of any uniform Hölder function. In the examples of Section 6, this upper bound proves to be the exact multifractal spectrum.

Definition 1. A function $f \in C^\varepsilon(\mathbb{R})$ for some $\varepsilon > 0$ is said to satisfy the multifractal formalism at the exponent $h \geq 0$ if $d_f(h) = (\xi_f)^*(h)$.

3. Multifractal formalism for measures

We consider a slight modification of the multifractal formalism developed in [16]. The main difference is located in the definition of the level sets E_α^μ. For our purpose, we only need the multifractal formalism associated with the dyadic grid of $[0, 1]$. Theorem 4 gives sufficient conditions for the validity of this formalism for measures built on the dyadic grid; Its proof is given in Section 7. We made this dyadic choice for sake of simplicity. Nevertheless, we mention that Theorem 4 also holds for measures that depend on a b-adic grid with b greater than 2 (see [12, 13]).

3.1 Hölder exponent, spectrum of singularity

Let us define $I_j^+(x) = I_j(x) + 2^{-j}$ and $I_j^-(x) = I_j(x) - 2^{-j}$. The convention $\log(0) = -\infty$ is again adopted.

Definition 2. Let μ be a positive Borel measure on $[0, 1]$. For $x_0 \in (0, 1)$, the lower and upper Hölder exponent of μ at x_0 are respectively defined by

$$\underline{\alpha}_\mu(x_0) = \liminf_{j\to+\infty} \frac{\log \mu(I_j(x_0))}{\log |I_j(x_0)|} \quad \text{and} \quad \overline{\alpha}_\mu(x_0) = \limsup_{j\to+\infty} \frac{\log \mu(I_j(x_0))}{\log |I_j(x_0)|}$$

When $\underline{\alpha}_\mu(x_0) = \overline{\alpha}_\mu(x_0)$, their common value is denoted $\alpha_\mu(x_0)$ and called the Hölder exponent of μ at x_0.

The left and right lower Hölder exponents of μ at x_0 are defined by

$$\underline{\alpha}_\mu^-(x_0) = \liminf_{j\to+\infty} \frac{\log \mu(I_j^-(x_0))}{\log |I_j^-(x_0)|} \quad \text{and} \quad \underline{\alpha}_\mu^+(x_0) = \liminf_{j\to+\infty} \frac{\log \mu(I_j^+(x_0))}{\log |I_j^+(x_0)|}.$$

We consider the following level sets for μ, that are necessary in our formalism

Definition 3. For every $\alpha \geq 0$, define

$$E_\alpha^\mu = \left\{x \in (0,1) \cap \text{supp}(\mu) : \alpha_\mu(x) = \alpha,\ \underline{\alpha}_\mu^-(x) \geq \alpha,\ \underline{\alpha}_\mu^+(x) \geq \alpha.\right\}. \quad (3.1)$$

The mapping $d_\mu : \ \alpha \geq 0 \mapsto \dim(E_\alpha^\mu)$ is called the multifractal spectrum of μ.

In the framework of [16], the level sets (3.1) are $\{x \in \text{supp}(\mu) : \alpha_\mu(x) = \alpha\}$. Unfortunately these simpler level sets are not adapted to our construction, since the knowledge of the sole exponent α_μ is not sufficient to guarantee the value of the pointwise Hölder exponent of the wavelet series F_μ.

3.2 Multifractal formalism for E_α^μ

Let μ be a positive Borel measure on $[0,1]$. For $j \geq 0$ and $k \in \mathbb{Z}$, let $I_{j,k}$ (*resp.* $I_{j,k}^+$ and $I_{j,k}^-$) denote the interval $[k2^{-j}, (k+1)2^{-j})$ (*resp.* $I_{j,k} + 2^{-j}$ and $I_{j,k} - 2^{-j}$). Following [16], let us define

$$\tau(q) = \liminf_{j\to+\infty} -\frac{1}{j} \log_2 \sum_{0\leq k\leq 2^j}^{*} \mu(I_{j,k})^q.$$

where $\sum^*$ means that the sum is taken over those k's such that $\mu(I_{j,k}) > 0$. The function τ is concave, non-decreasing. An equivalent definition for $\tau(q)$ is $\tau(q) = \sup\{t : \limsup_{j\to\infty} C_j(q,t) = 0\}$, where

$$C_j(q,t) = \sum_{0\leq k\leq 2^j}^{*} \mu(I_{j,k})^q 2^{jt}. \quad (3.2)$$

Noting that E_α^μ is always included in $\{x \in (0,1) : \alpha_\mu(x) = \alpha\}$, it follows from [16] that an upper bound for $\dim(E_\alpha^\mu)$ can be derived from the Legendre transform of τ. Recall the definition of the Legendre transform (2.4).

Proposition 1 Upper bound for $\dim(E^\mu_\alpha)$***.*** *Let* $\alpha \geq 0$*. One has*

$$\dim(E^\mu_\alpha) \leq \tau^*(\alpha).$$

Moreover, if $\tau^*(\alpha) < 0$ *then* $E^\mu_\alpha = \emptyset$*.*

Definition 4. The measure μ is said to obey the multifractal formalism at $\alpha \geq 0$ if $\dim(E^\mu_\alpha) = \tau^*(\alpha)$.

Remark 2. The formalism we use can be improved by considering the sets

$$\widetilde{E}^\mu_\alpha = \{x : \min(\underline{\alpha}_\mu(x), \underline{\alpha}^-_\mu(x), \underline{\alpha}^+_\mu(x)) = h_\mu(x) = \alpha\} \tag{3.3}$$

instead of the sets E^μ_α. Indeed, Proposition 1 also holds for these sets. It is this improved formalism which makes Theorem 1 applicable to the measure ν defined in (1.5) and the associated function F_ν.

Nevertheless we chose the more restrictive Definition 3 to ensure some stability properties after perturbations of wavelet coefficients in (see Section 5).$\widetilde{E}^\mu_\alpha$ instead of the E^μ_α.

Remark 3. Other multifractal formalisms involve simultaneous information on the quantities $\mu(I_j(x)^-)$, $\mu(I_j(x))$, and $\mu(I_j(x)^+)$. In [51] and [54], in order to define a grid-free multifractal formalism for the large deviation spectrum of μ, the function τ is derived from partition functions involving, instead of $\mu(I_{j,k})$, the μ-measure of the boxes $B^+_{j,k} = I^-_{j,k} \cup I_{j,k} \cup I^+_{j,k}$ such that $\mu(I_{j,k}) \neq 0$.

3.3 A sufficient condition of validity

The following theorem gives sufficient conditions for the validity of the multifractal formalism at a given point. Its proof is postponed in Section 7. Theorem 4 applies in particular to standard classes of statistically self-similar measures that may strongly depend on the dyadic grid with $b \geq 3$. Examples of measures are given in Section 6.

Let $\mathcal{A} = \{0, 1\}$. For every $w \in \mathcal{A}^* = \cup_{j \geq 0} \mathcal{A}^j$ ($\mathcal{A}^0 := \{\emptyset\}$), let I_w be the closed b-adic subinterval of $[0, 1]$ naturally encoded by w.

If $w \in \mathcal{A}^j$, one can assign to w a unique number $i(w)$ such that the interval I_w can be written $[i(w)2^{-j}, (i(w)+1)2^{-j})$. Then, if $(v, w) \in \mathcal{A}^j$, $\delta(v, w)$ stands for $|i(v) - i(w)|$.

Given $q \in \mathbb{R}$, a positive Borel measure μ_q, and a function C_q on $\mathcal{A}^*$ such that

$$\mu_q(I_w) \leq C_q(w) \mu(I_w)^q \, 2^{|w| \tau(q)} \quad \text{for all } w \in \mathcal{A}^* \text{ such that } \mu(I_w) > 0 \tag{3.4}$$

holds, if $\tau'(q)$ exists, one defines for $\varepsilon, \eta > 0$

$$S_1^{\mu}(q,\varepsilon,\eta) = \sum_{j\geq 1} 2^{j(\tau(q)+(\tau'(q)-\varepsilon)\eta)} \sum_{\substack{v,w\in\mathcal{A}^j:\,\delta(v,w)\leq 1,\\ \mu(I_w)>0}} \mu(I_v)^{\eta} C_q(w)\mu(I_w)^q$$

$$S_2^{\mu}(q,\varepsilon,\eta) = \sum_{j\geq 1} 2^{j(\tau(q)-(\tau'(q)+\varepsilon)\eta)} \sum_{w\in\mathcal{A}^j,\,\mu(I_w)>0} C_q(w)\mu(I_w)^{q-\eta}.$$

Definition 5. If ν is a positive Borel measure on $[0,1]$, its lower Hausdorff dimension is defined by $\dim(\nu) = \inf\{\dim(B) :\ \nu(B) > 0\}$.

Theorem 4. *Let μ be a positive Borel measure on $[0,1]$ and let $q \in \mathbb{R}$. Suppose that:*
(i) there exists a positive Borel measure μ_q on $[0,1]$ and a function C_q on $\mathcal{A}^*$ such that (3.4) holds.
(ii) $\tau'(q)$ exists, and $\dim(\mu_q) \geq \tau^*(\tau'(q))$.
(iii) for all $\varepsilon > 0$, there exists $\eta > 0$ such that $S_1^{\mu}(q,\varepsilon,\eta) + S_2^{\mu}(q,\varepsilon,\eta) < \infty$.
Then μ obeys the multifractal formalism at $\tau'(q)$, i.e. $\dim(E^{\mu}_{\tau'(q)}) = \tau^*(\tau'(q))$.

Remark 4. Condition *(iii)*, involving dyadic intervals and their neighbors, is comparable to the one provided in [8] for a measure satisfying the multifractal formalism of [16] for a dyadic grid to also satisfy the "centered" multifractal formalism [48].

4. Building multifractal wavelet series

In all the following sections, two real numbers $s_0 > 0$ and $p_0 > 0$ are fixed such that $s_0 - 1/p_0 > 0$. These parameters are used to specify the Besov spaces $B_p^{s,\infty}(\mathbb{R})$ the functions belong to.

4.1 Explicit construction based on measures

Definition 6. Let μ be a positive measure on $\mathbb{R}$. One defines the wavelet series F_μ, derived from μ, by the following formula

$$F_\mu(x) = \sum_{j\geq 0}\sum_{k\in\mathbb{Z}} d^{\mu}_{j,k}\psi_{j,k}(x), \tag{4.1}$$

where the wavelet coefficients $d^{\mu}_{j,k}$ are

$$d^{\mu}_{j,k} = 2^{-j(s_0-1/p_0)}\mu(I_{j,k})^{1/p_0}\,\sigma_{j,k}, \tag{4.2}$$

and where $\sigma_{j,k} \in \{-1,1\}$, and by convention $|0|^{1/p_0} = 0$.

Remark that the term $2^{-j(s_0-1/p_0)}$ ensures a minimal decay rate of the wavelet coefficients. Any C^∞ wavelet ψ can be used. As a consequence, several functions are built up here, depending on the choice of ψ. Nevertheless, Theorem 1 asserts that, under suitable assumptions on μ, these functions have the same multifractal spectrum. One can also consider compactly supported wavelets, but with the restrictions and the modifications we mentioned before (Section 2.1).

For every $m \in \mathbb{Z}$, let

$$F_{\mu,m}(x) = \sum_{j\geq 0} \sum_{m/2 \leq k2^{-j} < m/2+1} d^{\mu}_{j,k} \psi_{j,k}\left(m/2 + x\right).$$

Remark that for every point $x \in \mathbb{R}$, there exists an integer m such that $x \in (m/2, m/2 + 1)$.

Let us introduce at this point one possible definition for Besov spaces. These functional spaces are an especially relevant frame to work with in the frame of multifractal analysis of functions [30], and to find natural random wavelet series with concave spectra in such spaces was the initial motivation of this work. Let us recall the characterization of Besov spaces on $\mathbb{R}$ by wavelet coefficients (where any C^∞ wavelet $\widetilde{\psi}$ with all its moments of positive order equal to 0 can be chosen for the decomposition): For $p, q, s > 0$,

$$f \in B^{s,q}_p(\mathbb{R}) \Leftrightarrow \Big(\sum_k |d_{j,k} 2^{j(s-1/p)}|^p\Big)^{1/p} = \varepsilon_j \text{ with } (\varepsilon_j)_{j\geq 0} \in l^q. \tag{4.3}$$

The reader can verify the following lemma

Lemma 1. *If μ is a positive finite measure, then for all $m \in \mathbb{Z}$, $F_{\mu,m} \in B^{s_0,\infty}_{p_0}(\mathbb{R})$. Moreover, $F_{\mu,m}$ is C^∞ outside* $[0, 1]$.

This implies that

$$\text{for all } h \geq 0,\ d_{F_\mu}(h) = \sup_{m\in\mathbb{Z}} d_{F_{\mu,m}}(h).$$

Remark 5. It is now worth noting that the multifractal formalism introduced in [31, 33, 32], and used in this paper, is related to a generalization of Besov spaces, namely the oscillation spaces. The introduction of those spaces supply some limitation of the multifractal formalism associated with Besov spaces and the related scaling function (see [30] for instance).

4.2 Transfer of multifractality theorem

We recall Theorem 1. It links the singularity spectrum of F_μ to the one of μ. This result applies on each function $F_{\mu,m}$, $m \in \mathbb{Z}$. Without loss of generality, we redefine μ as its restriction to $[0, 1]$, and F_μ is now defined by

$$F_\mu(x) = \sum_{j\geq 0} \sum_{0\leq k<2^j} \sigma_{j,k} 2^{-j(s_0-1/p_0)} |\mu(I_{j,k})|^{1/p_0} \psi_{j,k}(x). \tag{4.4}$$

Theorem 1 *Let μ be a positive Borel measure, and let s_0, p_0 be two positive real numbers such that $s_0 - 1/p_0 > 0$. Let us consider the wavelet series (4.4). If μ obeys the multifractal formalism for measures at $\alpha \geq 0$, then F_μ obeys the multifractal formalism for functions at $h = s_0 - 1/p_0 + \alpha/p_0$, and*

$$d_{F_\mu}(h) = d_\mu(\alpha).$$

Moreover, one has $\xi_{F_\mu}(p) = p(s_0 - 1/p_0) + \tau(p/p_0)$, $\forall\, p \in \mathbb{R}$.

Remark 6. The regularity of the wavelet series F_μ in the complementary of the support of μ is governed only the one of the wavelet ψ. Indeed, if $x_0 \notin \operatorname{supp}(\mu)$, there exists j_x such that if $j \geq j_x$, $\mu(I_j(x)) = \mu(I_j^+(x)) = \mu(I_j^-(x)) = 0$, thus $d_{j,k} = 0$ for every couple (j,k) with $|k2^{-j} - x| \leq 2^{-j_x}$.

Proof. Let us focus on one couple (j,k), and on the definition of the wavelet coefficient $d^\mu_{j,k}$ of F_μ. In view of formula (4.2), it is obvious that for any couple (j',k') such that $I_{j',k'} \subset I_{j,k}$, $|d^\mu_{j',k'}| \leq | d^\mu_{j,k}|$. Hence, for any couple (j,k), if $L^\mu_{j,k}$ denotes the wavelet leader associated with F_μ, one has $L^\mu_{j,k} = |d^\mu_{j,k}|$, i.e. the wavelet coefficients coincide with the wavelet leaders.

Let $x_0 \in E^\mu_\alpha$. As a consequence of the previous remark, for every $j \geq 0$, one has $L_j(x_0) = 2^{-j(s_0 - 1/p_0)} \max(\mu(I_{j,k_{j,x_0}-1}), \mu(I_{j,k_{j,x_0}}), \mu(I_{j,k_{j,x_0}+1}))^{1/p_0}$. A direct application of Theorem 2 implies that $x_0 \in E^{F_\mu}_{s_0 - 1/p_0 + \alpha/p_0}$. In the rest of the proof we set $h = s_0 - 1/p_0 + \alpha/p_0$. Hence $E^\mu_\alpha \subset E^{F_\mu}_h$.

Assume that μ obeys the multifractal formalism for measures at α. Then, by definition of the scaling function τ associated with μ, one has $\tau^*(\alpha) = \dim E^\mu_\alpha \leq \dim E^{F_\mu}_h = d_{F_\mu}(h)$.

To get the upper bound for $d_{F_\mu}(h)$, we apply Theorem 3. Let us compute the scaling function ξ_{F_μ} associated with the wavelet series F_μ. Let $p \in \mathbb{R}$. One has (by convention $0^p = 0$ for all p)

$$\begin{aligned} \frac{\log \sum_{k=0}^{2^j-1} |L_{j,k}|^p}{\log 2^{-j}} &= \frac{\log \sum_{k=0}^{2^j-1} |d_{j,k}|^p}{\log 2^{-j}} = \frac{\log \sum_{k=0}^{2^j-1} 2^{-jp(s_0-1/p_0)} \mu(I_{j,k})^{p/p_0}}{\log 2^{-j}} \\ &= p(s_0 - 1/p_0) + \frac{\log \sum_{k=0}^{2^j-1} \mu(I_{j,k})^{p/p_0}}{\log 2^{-j}}. \end{aligned}$$

Thus $\xi_{F_\mu}(p) = p(s_0 - 1/p_0) + \tau(p/p_0)$, $\forall\, p \in \mathbb{R}$. Theorem 3 then implies that

$$\begin{aligned} d_{F_\mu}(h) &\leq (\xi_{F_\mu})^*(h) = \inf_{p \in \mathbb{R}} (ph - (p(s_0 - 1/p_0) + \tau(p/p_0))) \\ &\leq \inf_{p \in \mathbb{R}} (\alpha p/p_0 - \tau(p/p_0))) = \tau^*(\alpha). \end{aligned}$$

This, combined with the converse inequality, yields that $d_{F_\mu}(h) = \tau^*(\alpha)$. □

Remark 7. The form of the wavelet series F_μ we build (i.e. deduced from a positive measure) induces a hierarchy between the wavelet coefficients that makes Theorem 1 hold. In particular, one could consider, instead of a measure μ, more general non-decreasing set functions, such as for example Choquet capacities [38], provided that they satisfy some multifractal formalism.

Remark 8. Theorem 1 remains valid if the sets E^μ_α used in the multifractal formalism are replaced by the sets $\widetilde{E}^\mu_\alpha$ (3.3). This is used in Section 8 to derive the multifractal spectrum of F_μ when μ is a stable Lévy measure. This second formalism is nevertheless hard to manipulate when adding perturbations in the wavelet coefficients.

Remark 9. When $p=1$, $\mu(I_{j,k})$ can be viewed as $\langle \mu, \phi(2^j.-k)\rangle$, where $\phi(x)=\mathbf{1}_{[0,1]}(x)$. The mapping $\mu \to F_\mu = \sum_{j,k}\langle \mu, \phi_{j,k}\rangle \psi_{j,k}$ is a linear regularization operator.

5. Perturbing the construction

A natural question is the stability, from a multifractal viewpoint, of the construction if a perturbation is introduced in F_μ's wavelet coefficients.

5.1 Principles

The perturbation we consider consists in multiplying the wavelet coefficients by the terms of a real sequence $(\pi(j,k))_{j\geq 0,\ 0\leq k\leq 2^j-1}$. As in Section 4.2, without loss of generality, consider the wavelet series F_μ (4.1) with coefficients $d^\mu_{j,k}$ (4.2). Let us define, whenever it exists,

$$F^{pert}_\mu(x) = \sum_{j\geq 0, 0\leq k<2^j} d^{pert}_{j,k}\psi_{j,k}(x) \quad \text{with} \quad d^{pert}_{j,k} = 2^{-j(s_0-1/p_0)}\mu(I_{j,k})^{1/p_0}\pi(j,k).$$

Let us begin, without proof, with an easy classical perturbation principle.

Lemma 2. *Assume that $F_\mu \in C^\alpha([0,1])$, and let $\beta \in]-\infty,\alpha)$. Consider the set of perturbation coefficients $\pi(j,k)=2^{\beta j}$, $\forall j, k$. The wavelet series F^{pert}_μ deduced from F_μ by $d^{pert}_{j,k}=\pi(j,k)d^\mu_{j,k}=2^{\beta j}d^\mu_{j,k}$ belongs to $C^{\alpha-\beta}([0,1])$ and $d_{F^{pert}_\mu}(h)=d_{F_\mu}(h+\beta)$ for all $h\geq 0$.*

We shall need the following properties and definitions.

Property $(\mathcal{P}_1)$: $\displaystyle\limsup_{j\to\infty}\frac{\max_{0\leq k\leq 2^j-1}\log|\pi(j,k)|}{j}\leq 0.$

Property $(\mathcal{P}_2)$: $\displaystyle\liminf_{j\to\infty}\frac{\min_{0\leq k\leq 2^j-1}\log|\pi(j,k)|}{j}\geq 0.$

Property $(\mathcal{P}_3)$: The set $\displaystyle T=\left\{x : \limsup_{j\to+\infty}\frac{\log|\pi(j,k_{j,x})|}{j}<0\right\}$ is empty.

Property ($\mathcal{P}_4(d)$): $0 \le d < 1$ and $\dim T \le d$.

Proposition 2. *Let μ be a positive Borel measure on $[0,1]$. Suppose that the perturbations $(\pi(j,k))_{j\ge 0, 0\le k\le 2^j-1}$ satisfy $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$. Then, for every $\varepsilon > 0$, F_μ^{pert} belongs to $B_{p_0}^{s_0-\varepsilon,\infty}$ and the two wavelet series F_μ and F_μ^{pert} have the same exponents at every point x_0 (hence $d_{F_\mu} \equiv d_{F_\mu^{pert}}$). Moreover, $\xi_{F_\mu^{pert}} \equiv \xi_{F_\mu}$.*

Proposition 2, except the identity of $\xi_{F_\mu^{pert}}$ and $\xi_{F_\mu^{pert}}$, holds in fact for the perturbation f^{pert} of any wavelet series $f = \sum_{j\ge 0}\sum_{k\in\{0,\dots,2^j-1\}} d_{j,k}\psi_{j,k}$ (with arbitrary wavelet coefficients), as soon as f belongs to $\bigcup_{\varepsilon>0} C^\varepsilon(\mathbb{R})$.

Proof. By construction, for any $\varepsilon > 0$ there exists a scale J_ε such that

$$\forall j \ge J_\varepsilon,\ \forall j,k,\ 2^{-\varepsilon j} \le |\pi(j,k)| \le 2^{\varepsilon j}. \tag{5.1}$$

Equation (5.1) clearly implies that F_μ^{pert} belongs to $B_{p_0}^{s_0-\varepsilon,\infty}$ (since $F_\mu \subset B_{p_0}^{s_0,\infty}$).

Notice now that if $I_{j',k'} \subset I_{j,k}$, one has $d_{j',k'} \le 2^{(j-j')(s_0-1/p_0)} d_{j,k}$. Consequently, due to (5.1), if $\{L_{j,k}^{pert}\}$ denotes the set of wavelet leaders of F_μ^{pert}, then for j large enough, $\forall k$, one has $L_{j,k}2^{-2\varepsilon j} \le L_{j,k}^{pert} \le L_{j,k}2^{2\varepsilon j}$. Theorem 2 then shows that for any x_0, $h_{F_\mu}(x_0) - 2\varepsilon \le h_{F_\mu^{pert}}(x_0) \le h_{F_\mu}(x_0) + 2\varepsilon$. One concludes by letting ε go to zero. The above inequalities between leaders also yields $\xi_{F_\mu^{pert}} \equiv \xi_{F_\mu}$. □

Proposition 3. *Let μ be a positive Borel measure on $[0,1]$. Suppose that the perturbations $(\pi(j,k))_{j\ge 0, 0\le k\le 2^j-1}$ satisfy $(\mathcal{P}_1)$ and $(\mathcal{P}_3)$. Then, $\forall \varepsilon > 0$, F_μ^{pert} belongs to $B_{p_0}^{s_0-\varepsilon,\infty}$. Moreover, $\forall\, \alpha \ge 0$, if $h = s_0 - 1/p_0 + \alpha/p_0$, $d_\mu(\alpha) \le d_{F_\mu^{pert}}(h)$.*

Proof. Let $\varepsilon > 0$. $(\mathcal{P}_1)$ implies the existence of a scale J_ε such that for every $j \ge J_\varepsilon$, one has $\pi(j,k) \le 2^{\varepsilon j}$. Using the same arguments as in last Proposition, one gets that for every x_0, $h_{F_\mu}(x_0) \le h_{F_\mu^{pert}}(x_0)$, i.e. the local regularity can only increase after perturbation of the wavelet coefficients.

Let now $\alpha \ge 0$ and $x_0 \in E_\alpha^\mu$. By definition, $\lim_{j\to+\infty} \frac{\log \mu(I_j(x_0))}{\log 2^{-j}} = \alpha$. Condition $(\mathcal{P}_3)$ implies that there are infinitely many scales j such that $\pi(j,k_{j,x_0}) \ge 2^{-\varepsilon j}$. Thus, if $\{L_{j,k}^{pert}\}$ denotes the set of wavelet leaders of F_μ^{pert}, then one has $L_j^{pert}(x_0) \ge 2^{-j(s_0-1/p_0)}\mu(I_j(x_0))^{1/p_0}2^{-2\varepsilon j}$ for infinitely many scales j. Then, Theorem 2 implies that $h_{F_\mu^{pert}}(x_0) \le h_{F_\mu}(x_0) + 2\varepsilon = s_0 - 1/p_0 + \alpha/p_0 + 2\varepsilon$, $\forall\, \varepsilon > 0$. Thus $E_\alpha^\mu \subset E_h^{F_\mu^{pert}}$, and $d_\mu(\alpha) = \dim E_\alpha^\mu \le \dim E_h^{F_\mu^{pert}} = d_{F_\mu^{pert}}(h)$. □

Let $\alpha_{\min} = \inf\{\alpha : \tau_\mu^*(\alpha) > 0\}$. We let the reader verify that $(\mathcal{P}_1)$ implies that whenever $p > 0$, $\xi_{F_\mu^{pert}}(p) \ge \xi_{F_\mu}(p)$. As a consequence of

Proposition 3 and Theorem 3, if μ obeys the multifractal formalism at every $\alpha \in (\alpha_{\min}, \tau'(0^+)]$ (i.e. in the increasing part of the spectrum d_μ), then F_μ^{pert} obeys the multifractal formalism at every $h \in (s_0-1/p_0+\alpha_{\min}/p_0, s_0-1/p_0+\tau'(0^+)/p_0]$.

Proposition 4. *Let μ be a positive Borel measure on $[0,1]$. Suppose that the perturbations $(\pi(j,k))_{j\geq 0, 0\leq k\leq 2^j-1}$ satisfy $(\mathcal{P}_1)$ and $(\mathcal{P}_4(d))$ for some $d\in[0,1)$. Then, the wavelet series F_μ^{pert} belongs to $B_{p_0}^{s_0-\varepsilon,\infty}$ for every $\varepsilon>0$. Moreover, for every $\alpha\geq 0$ such that $d_\mu(\alpha)>d$, one has $d_\mu(\alpha)\leq d_{F_\mu^{pert}}(h)$ where $h=s_0-1/p_0+\alpha/p_0$.*

Proof. $(\mathcal{P}_4(d))$ replaced $(\mathcal{P}_3)$, so the second argument in the proof of Proposition 3 holds at every $x\notin T$. This is enough to conclude. □

Remark 10. Remark that no more hierarchical relation between the wavelet coefficients holds after multiplication of $d_{j,k}$ by $\pi(j,k)$. However our analysis shows that $d_{F_\mu^{pert}}$ can be computed for some values of h.

5.2 Random perturbations

We give sufficient conditions for properties $(\mathcal{P}_i)$ to hold almost surely if the sequence $(\pi(j,k))_{j\geq 0,0\leq k\leq 2^j-1}$ is a sequence of real random variables.

Proposition 5. *Sufficient conditions for perturbations:*

- $(\mathcal{P}_1)$ *holds if* $\lim_{q\to\infty}\frac{1}{q}\limsup_{j\to\infty}\frac{\max_{0\leq k\leq 2^j-1}\log\mathbb{E}(|\pi(j,k)|^q)}{j}=0.$
- $(\mathcal{P}_2)$ *holds if* $\sum_{j\geq 0}2^j\max_{0\leq k\leq 2^j-1}\mathbb{P}(\pi(j,k)=0)<\infty$ *and*

$$\lim_{q\to-\infty}\frac{1}{|q|}\limsup_{j\to\infty}\frac{\max_{0\leq k\leq 2^j-1}\log\,\mathbb{E}(\mathbf{1}_{\{\pi(j,k)\neq 0\}}|\pi(j,k)|^q)}{j}=0.$$

- $(\mathcal{P}_3)$ *holds if the random variables $\pi(j,k)$ are independent and if for every $\varepsilon>0$, $\lim_{j\to\infty}\max_{0\leq k\leq 2^j-1}\mathbb{P}(|\pi(j,k)|\leq 2^{-j\varepsilon})=0$.*
- $(\mathcal{P}_4(d))$ *holds if the random variables $\pi(j,k)$ are independent and if $\forall\varepsilon>0$,*

$$\limsup_{j\to+\infty}\max_{0\leq k\leq 2^j-1}\mathbb{P}(|\pi(j,k)|\leq 2^{-j\varepsilon})\leq 2^{d-1}.$$

Proof. We begin by
- $(\mathcal{P}_2)$: Fix $\varepsilon>0$. For all $q<0$ and $j\geq 0$,

$$\begin{aligned}&\mathbb{P}(\exists\ 0\leq k\leq 2^j-1: 0\leq|\pi(j,k)|\leq 2^{-j\varepsilon})\\ &\leq\sum_{k=0}^{2^j-1}\mathbb{P}(\pi(j,k)=0)+\mathbb{P}(\pi(j,k)\neq 0,|\pi(j,k)|^q>2^{-jq\varepsilon})\\ &\leq 2^j\Big(\max_{0\leq k\leq 2^j-1}\mathbb{P}(\pi(j,k)=0)+\max_{0\leq k\leq 2^j-1}2^{jq\varepsilon}\mathbb{E}(\mathbf{1}_{\{\pi(j,k)\neq 0\}}|\pi(j,k)|^q)\Big).\end{aligned}$$

Fix $\varepsilon' > 0$, $j_0 \geq 0$ and $q < 0$ such that $\alpha = 1 + q\varepsilon + |q|\varepsilon' < 0$ and

$$\forall j \geq j_0,\ j^{-1} \max_{0 \leq k \leq 2^j - 1} \log \mathbb{E}(\mathbf{1}_{\{\pi(j,k) \neq 0\}} |\pi(j,k)|^q) \leq |q|\varepsilon'.$$

Then $\sum_{j \geq j_0} 2^j \max_{0 \leq k \leq 2^j-1} 2^{jq\varepsilon} \mathbb{E}(\mathbf{1}_{\{\pi(j,k) \neq 0\}} |\pi(j,k)|^q) \leq \sum_{j \geq j_0} 2^{j\alpha} < +\infty$. Using that $\sum_{j \geq 0} 2^j \max_{0 \leq k \leq 2^j-1} \mathbb{P}(\pi(j,k) = 0) < +\infty$, by applying the Borel-Cantelli Lemma , one obtains that with probability one, there exists an integer J such that, $\forall j \geq J$, $\forall 0 \leq k \leq 2^j - 1$, $|\pi(j,k)| > 2^{-j\varepsilon}$. The conclusion follows after letting ε tend to 0 along a countable sequence.

- ($\mathcal{P}_1$): The same proof as for ($\mathcal{P}_2$) holds.

- ($\mathcal{P}_3$): For every $\varepsilon > 0$, let us define $T_\varepsilon = \{x \in [0,1] : \exists J,\ \forall j \geq J,\ |\pi(j, k_{j,x})| \leq 2^{-j\varepsilon}\}$. One remarks that $T_\varepsilon \subset \bigcup_{J \geq 0} U_J$, where

$$U_J = \bigcap_{j \geq J} \bigcup_{0 \leq k \leq 2^j - 1:\, |\pi(j,k)| \leq 2^{-j\varepsilon}} I_{j,k}.$$

Each U_J is the boundary of a dyadic branching tree in a random environment with extinction probability $1 - \mathbb{P}(|\pi(j,k)| \leq 2^{-j\varepsilon})$ (which tends to 1 uniformly in k when $j \to +\infty$) at node indexed by (j,k). Since the random variables $\pi(j,k)$ are mutually independent, T_ε is almost surely empty. Indeed, for j large enough, the probability of extinction of one single node of the j^{th} generation becomes larger than the one in a sub-critical Galton-Watson subtree of $\{0,1\}^*$. Eventually, $T \subset \bigcup_{n \geq 1} T_{1/n}$, thus T is empty with probability one.

- ($\mathcal{P}_4(d)$): In the sense of [21] (see the percolation results therein), for every $\varepsilon > 0$, the set T_ε is included in the set of "bad paths" in $\{0,1\}^{\mathbb{N}}$, where every node is "bad" with probability $P_{j,\varepsilon} = \max_{0 \leq k \leq 2^j-1} \mathbb{P}(|\pi(j,k)| \leq 2^{-j\varepsilon})$ and "good" with $1 - P_{j,\varepsilon}$, a node being "good" or "bad" independently of the other nodes. It follows that

$$\text{with probability one,} \quad \dim T_\varepsilon \leq 1 - \limsup_{j \to +\infty} j^{-1} \sum_{k=1}^{j} \log_2(1/P_{j,\varepsilon}).$$

One concludes again by writing $T \subset \bigcup_{n \geq 1} T_{1/n}$. □

5.3 Examples

- Uniform control on $\pi(j,k)$: ($\mathcal{P}_1$) (resp. ($\mathcal{P}_2$)) holds almost surely if the $\pi(j,k)$ are identically distributed with a random variable with finite moments of every positive (resp. negative) order. This is used in Section 6.2.

- Gaussian $\pi(j,k)$: ($\mathcal{P}_1$) and ($\mathcal{P}_3$) hold almost surely simultaneously if the $\pi(j,k)$ are independent centered Gaussian random variables with variance $\sigma(j,k)$ such that $\lim_{j \to \infty} j^{-1} \max_{0 \leq k \leq 2^j-1} |\log \sigma(j,k)| = 0$. This makes it possible the construction of Gaussian processes with prescribed Hausdorff multifractal spectrum (via a multifractal measure). This principle works with the examples of Section 6 (of course, for a random measure μ, the Gaussian perturbations have to be chosen independently of μ).

It also allows to construct very easily "pseudo" Fractional Brownian Motion (FBM) in the following sense. Let us fix $(s_0, p_0) = (2, 1)$, and let us then consider the wavelet series F_ℓ^{pert}, where ℓ is the Lebesgue measure, and where the perturbations $\pi(j,k)$ are as above. Consequently, for every $H \in (0,1)$, due to Lemma 2, the wavelet series F_H deduced from F_ℓ^{pert} by

$$d_{j,k}(F_H) = 2^{(2-H)j} d_{j,k}(F_\ell^{pert}) = 2^{-jH}\pi(j,k)$$

is a Gaussian process on $[0,1]$. Its multifractal spectrum is the same as the one of the FBM of exponent H, i.e. $d_{F_H}(H) = 1$, and $d_{F_H}(h) = -\infty$ if $h \neq H$ (note that the low frequency part of the FBM is forgotten here). Such a method has been already considered in [23]. For a complete construction of the FBM with exponent H based on wavelet coefficients, see [45].

- Lacunary $\pi(j,k)$: Fix $d \in [0,1]$. $(\mathcal{P}_1)$ and $(\mathcal{P}_4(d))$ hold almost surely if the $\pi(j,k)$ are i.i.d random variables that take the value 0 with probability 2^{d-1} and 1 with probability $1-2^{d-1}$. If μ obeys the multifractal formalism on $(\alpha_{\min}, \tau'(0^+)]$ the perturbation operation produces lacunary wavelet series F_μ^{pert}, with a multifractal spectrum equal to the one of F_μ at every $h = s_0 - 1/p_0 + \gamma/p_0$ such that $\gamma \in (\alpha_{\min}, \tau'(0^+)]$ and $\tau^*(\gamma) > d$.

Lacunary wavelet series are also considered in [29]. They correspond to perturbations of the function F_ℓ where $s_0 = 1 + \alpha$ and $p_0 = 1$ for some $\alpha \in (0,1)$. But the way certain wavelet coefficients of generation j are killed is different. For some fixed $\eta \in (0,1)$, at each scale j, $[2^{j\eta}]$ wavelet coefficients are selected uniformly and independently. These selection processes are mutually independent. The non-selected coefficients are put to 0. A major difference with our perturbation process is that there, with probability one, the pointwise Hölder exponent is modified on a set of full Lebesgue measure, and is left unchanged on a set of Hausdorff dimension equal to η. This also gives rise to interesting spectra.

6. Wavelet series derived from statistically self-similar measures

In the following examples, when a measure μ is defined on $(I, \mathcal{B}(I))$ with $I \in \{[0,1], \mathbb{R}_+\}$, in order to define F_μ, we implicitly consider on $\mathbb{R}$ the extension of μ by setting $\mu = 0$ outside I, and without ambiguity we say that μ (resp. F_μ) obeys the multifractal formalism whenever the redefined measure (resp. function) as defined in Section 4.2 obeys it.

Most of the classical families of statistically self-similar measures satisfy the conditions of Theorem 4. For instance, Quasi-Bernoulli measures [16, 20, 22, 35], Mandelbrot cascades [40], compound Poisson cascade measures [9] and stable Lévy measures [14] belong to this class.

In this work, we detail the example of dyadic Mandelbrot random multiplicative cascades, and we refer the reader to [12, 13] for more details on statistical self-similar measures and for the proof of Theorem 4 in these cases. The wavelet series F_μ associated with Mandelbrot cascades is particularly interesting, since the perturbations of such series allows us to derive the Hausdorff multifractal spectrum of the "random wavelet cascades" of Arnéodo, Bacry and Muzy [3]. We also give some clues of what happens when considering stable Lévy measures. The proofs of next Theorems 5 and 6 are postponed to Section 8.

6.1 Dyadic random multiplicative cascades

We consider the random "canonical" cascade measures introduced by B. Mandelbrot in [40, 41]. Their analysis led to a large literature [37, 36, 26, 18, 46, 7, 6, 49].

Let us fix a positive random variable W. We assume that W is not almost surely constant and that $\mathbb{E}(W) = 1/2$. Let us introduce the function

$$q \in \mathbb{R} \mapsto \widetilde{\tau}(q) = -1 - \log_2 \mathbb{E}(\mathbf{1}_{\{W>0\}} W^q). \tag{6.1}$$

In order to avoid technicalities, unessential to our purpose, we assume in this section that W is positive and that $\widetilde{\tau}(q) > -\infty$ for all $q \in \mathbb{R}$.

Let $(W_w)_{w\in\mathcal{A}^*}$ be a sequence of independent copies of W. For every $n \geq 1$, let us consider the random measure μ_n on $\mathbb{R}$ with density with respect to the Lebesgue measure given on every interval I_w, $w = w_1 w_2 \dots w_n$, by

$$2^n W_{w_1} W_{w_1 w_2} \dots W_{w_1 w_2 \dots w_n}$$

and such that $\mu_n = 0$ outside $[0, 1]$ (see Section 3.3 for the definition of I_w). With probability one, the sequence μ_n converges vaguely to a measure μ when n goes to infinity. Moreover, if $\widetilde{\tau}'(1) > 0$, one has $\mu \neq 0$ with positive probability [37]. Since W is chosen positive, $\mu \neq 0$ with probability one [26].

Then, let us introduce the set $\mathcal{J} = \{q \in \mathbb{R};\ \widetilde{\tau}^*(\widetilde{\tau}'(q)) > 0\}$. It follows from Theorem 8(iv) in [6] that $\tau = \widetilde{\tau}$ on $\mathcal{J}$.

Theorem 5. *Let μ be a dyadic random multiplicative cascade. With probability one, for every $q \in \mathcal{J}$, the associated wavelet series F_μ obeys the multifractal formalism at $h = s_0 - 1/p_0 + \widetilde{\tau}'(q)/p_0$ and $d_{F_\mu}(h) = \widetilde{\tau}'(q)q - \widetilde{\tau}(q)$. Moreover, $E_h^{F_\mu} \cap (0,1) = \emptyset$ for all $h \notin \overline{\{s_0 - 1/p_0 + \widetilde{\tau}'(q)/p_0 :\ q \in \mathcal{J}\}}$.*

Remark 11. **1**. If $\mathbb{P}(W=0)>0$, the same kind of conclusion holds, but in certain cases the interval $\mathcal{J}$ has to be reduced. This is due to the non-existence of certain moment of negative orders of $\mu([0,1])$ conditionally on the fact that $\mu \neq 0$ (see [6] Remark 1 and Theorem 8$(i)(b)$ for more details).

2. In certain cases this result can be completed by using some results of [6] on the endpoints of $\mathcal{J}$. We differ the use of these results to the application to [3] below.

6.2 The natural perturbation and an application to [3]

It turns out from the definition of μ that $\forall w \in \mathcal{A}^*$, there exists a copy $Y(w)$ of $\mu([0,1])$ such that

$$\mu(I_w) = Y(w)\mu_{|w|}(I_w).$$

This reflects what we call the statistical self-similarity.

Moreover, if $W \leq 1$ and $\mathbb{P}(W=1) < 1/2$, all the moments of $\mu([0,1])$ are finite (see [36, 46, 7] for moments of negative orders and [37] for moments of positive orders). Consequently, if we consider the perturbations coefficients

$$\pi(j,k) = \left(\frac{\mu_j(I_{j,k})}{\mu(I_{j,k})}\right)^{1/p_0}, \tag{6.2}$$

we are in the context of the first example of Section 5.2. As a consequence, the conclusions of Theorem 5 hold for F_μ^{pert} instead of F_μ.

In [3], a random variable $\mathcal{W}$ is chosen with the following properties: $\mathbb{P}(|\mathcal{W}|>0)=1$, $-\infty < \mathbb{E}(\log|\mathcal{W}|) < 0$, and there exists $\eta > 0$ such that for every $h \in [0,\eta]$,

$$f(h) = \inf_{q\in\mathbb{R}} \left(hq + 1 + \log_2 \mathbb{E}(|\mathcal{W}|^q)\right) < 0. \tag{6.3}$$

Then, a sequence $(\mathcal{W}_w)_{w\in\mathcal{A}^*}$ of independent copies of $\mathcal{W}$ is chosen, and a random wavelet series F is defined by its wavelet coefficients as follows (w is such that $I_{j,k} = I_w$)

$$d_{j,k}(F) = \mathcal{W}_{w_1}\mathcal{W}_{w_1w_2}\dots\mathcal{W}_{w_1w_2\dots w_j}.$$

It can be seen that F converges almost surely in $L^2((0,1))$.

By using large deviations results, in [3] the authors show that the pointwise Hölder exponents of F belong to the interval $[h_{\min}, h_{\max}]$ where $h_{\min} = \inf\{0 < h < -\mathbb{E}(\log_2 \mathcal{W}) : \ f(h) \geq 0\}$ and $h_{\max} = \inf\{h > -\mathbb{E}(\log \mathcal{W}) : \ f(h) < 0\}$. Moreover, with probability one, for every $\alpha \in (0, h_{\min})$, $F \in C^\alpha((0,1))$.

We claim that in certain cases the series F can be viewed as a perturbation of one wavelet series F_μ associated with a suitable dyadic random multiplicative cascade measure μ. As a consequence, when this holds, the Hausdorff multifractal spectrum of F can be computed.

Let us assume that **all the moments of $\mathcal{W}$ are finite**. We consider

$$T: \ q \in \mathbb{R} \mapsto -1 - \log_2 \mathbb{E}(|\mathcal{W}|^q) \quad \text{and} \quad W = \frac{|\mathcal{W}|}{2\mathbb{E}(|\mathcal{W}|)}$$

With W can be associated the scaling function $\widetilde{\tau}$ (6.1) and the interval $\mathcal{J}$. For every $q \in \mathbb{R}$, one easily sees that $T(q) = -q(1 + \log_2 \mathbb{E}(|\mathcal{W}|)) + \widetilde{\tau}(q)$. Hence one has

$$T'(q) = -1 - \log_2 \mathbb{E}(|\mathcal{W}|) + \widetilde{\tau}'(q) \quad \text{and} \quad f(T'(q)) = T^*(T'(q)) = \widetilde{\tau}^*(\widetilde{\tau}'(q)). \tag{6.4}$$

This implies that $(h_{\min}, h_{\max}) = \{T'(q) : \ q \in \mathcal{J}\}$.

Moreover, using (6.3) and (6.4), one gets $\widetilde{\tau}^*(\widetilde{\tau}'(q)) < 0$ for some $q > 0$. This implies that $\widetilde{\tau}$ becomes positive at 1^+ and that $\widetilde{\tau}'(1) > 0$.

As a consequence, let us consider the measure μ constructed as in the previous Section 6.1 with the weights $\{W_{w_1 \dots w_n}\}_{w \in \mathcal{A}^*} = \left\{ \frac{|\mathcal{W}_{w_1 \dots w_n}|}{2\mathbb{E}(|\mathcal{W}|)} \right\}_{w \in \mathcal{A}^*}$. Let us then introduce the corresponding wavelet series F_μ with parameters $s_0 = 2$ and $p_0 = 1$ (with coefficients $d_{j,k}(F_\mu)$), and its perturbation F_μ^{pert} built with the real sequence (6.2) (with coefficients $d_{j,k}(F_\mu^{pert})$). One can then rewrite the wavelet coefficients of the wavelet series F as ($w = w_1 \dots w_j$ is chosen so that $I_{j,k} = I_{w_1 \dots w_j}$)

$$\begin{aligned} |d_{j,k}(F)| &= 2^{(s_0 - 1/p_0)j} \big(2\mathbb{E}(|\mathcal{W}|)\big)^j 2^{-(s_0 - 1/p_0)j} W_{w_1} \dots W_{w_1 \dots w_j} \\ &= 2^{(2 + \log_2 \mathbb{E}(|\mathcal{W}|))j} |d_{j,k}(F_\mu^{pert})|. \end{aligned}$$

In order to use the result on F_μ^{pert}, we assume that $W \leq 1$ and $\mathbb{P}(W = 1) < 1/2$. We then apply Theorem 5 and Lemma 2. With probability one, for every $q \in \mathcal{J}$, at point

$$h_q = -(2 + \log_2 (\mathbb{E}(|\mathcal{W}|)) + (s_0 - 1/p_0 + \widetilde{\tau}'(q)/p_0) = T'(q),$$

one has $d_F(h_q) = \widetilde{\tau}'(q) q - \widetilde{\tau}(q) = f(h_q)$ ((6.4) has been used here). Moreover, $E_h^F = \emptyset$ for every $h \notin \overline{\{T'(q) : \ q \in \mathcal{J}\}}$.

Corollary 1. *Under the above assumptions on $\mathcal{W}$, with probability one, one has $d_F(h) = f(h)$ for every $h \in (h_{\min}, h_{\max})$.*
Moreover, $E_h^F = \emptyset$ for every $h \notin [h_{\min}, h_{\max}]$.

Remark 12. The results concerning the endpoints $h_{\min}$ and $h_{\max}$ can be completed using [6]. Moreover, if $h_{\min} = T'(q_{\max})$ and $h_{\max} = T'(q_{\min})$, one can use the results of [18, 49] to specify ξ_F outside $[q_{\min}, q_{\max}]$.

Corollary 2. *Under the assumptions of Corollary 1, let us suppose the additional properties on $\mathcal{W}$:*

(i) $\mathbb{R}_+ \subset \mathcal{J}$ *and* $(\widetilde{\tau})^*(\alpha_{\min}) > 0$*, or* $\mathbb{R}_+ \not\subset \mathcal{J}$*,*
(ii) $\mathbb{R}_- \subset \mathcal{J}$ *and* $(\widetilde{\tau})^*(\alpha_{\max}) > 0$*, or* $\mathbb{R}_- \not\subset \mathcal{J}$*.*

Then, with probability one,

$$d_F(h) = \begin{cases} f(h) = (\xi_F)^*(h) & \text{if } h \in [h_{\min}, h_{\max}] \\ -\infty & \text{otherwise,} \end{cases}$$

$$\text{and} \quad \xi_F(p) = \begin{cases} h_{\max}\, p & \text{if } -\infty < q_{\min} \text{ and } p \le q_{\min} \\ -\log_2 \mathbb{E}(|\mathcal{W}|^p) & \text{if } p \in \mathcal{J} \\ h_{\min}\, p & \text{if } q_{\max} < \infty \text{ and } p \ge q_{\max}. \end{cases}$$

In particular, F *obeys the multifractal formalism on* $[h_{\min}, h_{\max}]$*.*

6.3 Stable Lévy measures

Fix $\beta \in (0,1)$. Let S_β be a Poisson point process in $\mathbb{R}_+ \times \mathbb{R}_+^*$ with intensity $\ell \otimes \nu$, where ℓ is the Lebesgue measure and $\frac{d\nu}{d\ell}(\lambda) = \lambda^{-1-\beta}$. The measure μ defined by

$$\mu = \sum_{(t,\lambda) \in S_\beta} \lambda.\delta_t$$

is almost surely finite and called β-stable Lévy measure. The process $X_\beta(t) = \mu([0,t])$ is a β-stable Lévy subordinator. The measure μ is statistically invariant by positive horizontal translations.

We focus on the stable case, which simplifies the computation of the function τ.

Theorem 6. *With probability one,* F_μ *obeys the multifractal formalism at every* $h \in [s_0 - 1/p_0, s_0 - 1/p_0 + 1/\beta p_0]$*, with* $d_{F_\mu}(h) = \beta p_0(h - s_0 + 1/p_0)$*.*

Moreover, $E_h^{F_\mu} \cap (0,1) = \emptyset$ *for all* $h \notin [s_0 - 1/p_0, s_0 - 1/p_0 + 1/\beta p_0]$*.*

7. Proof of Theorem 4

Theorem 4 is a consequence of the next proposition and two lemmas. In the sequel, the assumptions of Theorem 4 are supposed to hold.

Lemma 3. $\min(\underline{\alpha}_\mu^-(x), \underline{\alpha}_\mu(x), \underline{\alpha}_\mu^+(x)) \ge \tau'(q)$ μ_q*-almost everywhere.*

Proof. For $j \ge 1$ and $\varepsilon > 0$, for $* \in \{-, 0, +\}$, let us define

$$\underline{E}_{j,\varepsilon}^* = \left\{ x \in \operatorname{supp}(\mu) : \frac{\log \mu(I_j(x)^*)}{\log |I_j(x)^*|} \le \tau'(q) - \varepsilon \right\},$$

with the convention $\underline{E}_{j,\varepsilon}^0 = \underline{E}_{j,\varepsilon}$ and $I_j(x)^0 = I_j(x)$.

Fix $\varepsilon, \eta > 0$, and let us focus on $\underline{E}_{j,\varepsilon}^-$. Let $Y(t)$ be the random variable equal to $2^{j(\tau'(q)-\varepsilon)\eta} \mu(I_j(t)^-)^\eta$. The Markov inequality applied to $Y(t)$ with respect to μ_q yields $\mu_q(\{t : Y(t) \geq 1\}) \leq \int Y(t) d\mu_q(t)$. Since Y is constant on any dyadic interval, one gets

$$\mu_q(\underline{E}^-_{j,\varepsilon}) \leq \sum_{0\leq k\leq 2^j-1} \mu_q(I_{j,k})\mu(I^-_{j,k})^\eta 2^{j\eta(\tau'(q)-\varepsilon)}.$$

Using (3.4) yields, with $C_q(j,k) = C_q(w)$ if $I_{j,k} = I_w$,

$$\mu_q(\underline{E}^-_{j,\varepsilon}) \leq 2^{j(\tau(q)+\eta(\tau'(q)-\varepsilon))} \sum_{0\leq k\leq 2^j-1,\, \mu(I_{j,k})>0} \mu(I^-_{j,k})^\eta C_q(j,k)\mu(I_{j,k})^q.$$

The same upper bound holds for $\mu_q(\underline{E}_{j,\varepsilon})$ and $\mu_q(\underline{E}^+_{j,\varepsilon})$, by replacing $\mu(I^-_{j,k})^\eta$ respectively by $\mu(I_{j,k})^\eta$ and $\mu(I^+_{j,k})^\eta$. Hence, for all $\varepsilon, \eta > 0$, for some constant $C(q,\varepsilon,\eta)$, one has

$$\sum_{j\geq 1} \mu_q\left(\underline{E}^-_{j,\varepsilon} \cup \underline{E}_{j,\varepsilon} \cup \underline{E}^+_{j,\varepsilon}\right) \leq C(q,\varepsilon,\eta) S^\mu_1(q,\varepsilon,\eta).$$

Choosing η such that $S^\mu_1(q,\varepsilon,\eta) < \infty$ yields $\sum_{j\geq 1} \mu_q\left(\underline{E}^-_{j,\varepsilon} \cup \underline{E}_{j,\varepsilon} \cup \underline{E}^+_{j,\varepsilon}\right) < \infty$ for all $\varepsilon > 0$ and the conclusion follows by the Borel-Cantelli Lemma. □

Lemma 4. *$\overline{\alpha}_\mu(x) \leq \tau'(q)$ μ_q-almost everywhere.*

Proof. For $j \geq 1$ and $\varepsilon > 0$, define

$$\overline{E}_{j,\varepsilon} = \left\{x \in \text{supp}(\mu) : \frac{\log \mu(I_j(x))}{\log |I_j(x)|} \geq \tau'(q) + \varepsilon\right\}.$$

Fix $\varepsilon, \eta > 0$. As above, it follows from the Markov inequality applied to the random variable $Y(t) = 2^{-j(\tau'(q)+\varepsilon)\eta}\mu(I_j(t))^{-\eta}$ that

$$\mu_q(\overline{E}_{j,\varepsilon}) \leq \sum_{0\leq k\leq 2^j-1:\, \mu(I_{j,k})>0} \mu_q(I_{j,k})\mu(I_{j,k})^{-\eta} 2^{-j\eta(\tau'(q)+\varepsilon)}.$$

This holds for every $j \geq 1$. Using (3.4), it follows that

$$\mu_q(\overline{E}_{j,\varepsilon}) \leq 2^{j\tau(q)-j(\tau'(q)+\varepsilon)\eta} \sum_{0\leq k\leq 2^j-1} C_q(j,k)\mu(I_{j,k})^{q-\eta}.$$

By summing the last inequality over all $j \geq 1$, one obtains that

$$\sum_{j\geq 1} \mu_q(\overline{E}_{j,\varepsilon}) \leq S^\mu_2(q,\varepsilon,\eta).$$

The conclusion then follows as in Lemma 3. □

Proof. (of Theorem 4) Due to Lemma 3 and 4, μ_q is carried by $E^\mu_{\tau'(q)}$. Consequently, our assumption on $\dim(\mu_q)$ implies that $\dim(E^\mu_{\tau'(q)}) \geq \tau^*(\tau'(q))$. Moreover, $\dim(E^\mu_{\tau'(q)})$ is always less than $\tau^*(\tau'(q))$. Thus $\dim(E^\mu_{\tau'(q)}) = \tau^*(\tau'(q))$ and the multifractal formalism holds at $\tau'(q)$. □

8. Proofs of Theorems 5 and 6

8.1 Dyadic random multiplicative cascades

Some computations use arguments that were already present in [8]. For every $q \in \mathcal{J}$, $v \in \mathcal{A}^*$ and $j \geq 1$, let

$$Y_{q,j}(v) = 2^{j\widetilde{\tau}(q)} \sum_{w_1\dots w_j \in \mathcal{A}^j} W^q_{vw_1} W^q_{vw_1w_2} \dots W^q_{vw_1w_2\dots w_j}.$$

It follows from Corollary 5 in [6] that, with probability one, for all $v \in \mathcal{A}^*$ and all $q \in \mathcal{J}$, the limit $Y_q(v) = \lim_{j\to\infty} Y_{q,j}(v)$ exists. Moreover, with probability one, for all $q \in \mathcal{J}$, the mapping μ_q defined on the dyadic intervals by

$$\mu_q(I_v) = 2^{|v|\widetilde{\tau}(q)} Y_q(v) \prod_{j=1}^{|v|} W^q_{v_1\cdots v_j} \tag{8.1}$$

extends to a Borel measure (notice that $\mu_1 = \mu$). All the measures μ_q have their support equal to $[0,1]$ and for all $v \in \mathcal{A}^*$ and $q \in \mathcal{J}$,

$$\mu_q(I_v) = C_q(v)\mu(I_v)^q 2^{|v|\widetilde{\tau}(q)} \tag{8.2}$$

with $C_q(v) = \frac{Y_q(v)}{Y_1^q(v)}$. Moreover, $\forall\, q \in \mathcal{J}$ one has $\dim(\mu_q) \geq \widetilde{\tau}^*(\widetilde{\tau}'(q)) = \tau^*(\tau'(q))$.

Since $\tau = \widetilde{\tau}$ on $\mathcal{J}$, the first part of Theorem 5 is a consequence of Theorem 1 and 4 if the following property holds: For every non-trivial compact subinterval K of $\mathcal{J}$, with probability one, for all $q \in K$, for all $\varepsilon > 0$, there exists $\eta > 0$ such that for $\gamma \in \{-1, 1\}$

$$\sum_{j\geq 1} 2^{j(\tau(q)+\gamma\eta\tau'(q)-\varepsilon\eta)} \sum_{v,w\in\mathcal{A}^j:\delta(v,w)\leq 1} \mu(I_v)^{\gamma\eta} C_q(w)\mu(I_w)^q < \infty.$$

This is equivalent to $\sum_{j\geq 1} 2^{j(\tau(q)+\gamma\eta\tau'(q)-\varepsilon\eta)} f_{j,\varepsilon,\eta}(q) < \infty$, where

$$f_{j,\varepsilon,\eta}(q) = \sum_{v,w\in\mathcal{A}^j:\, \delta(v,w)\leq 1} Y_1(v)^{\gamma\eta} Y_q(w) \prod_{k=1}^{j} W^{\gamma\eta}_{v_1\dots v_k} W^q_{w_1\dots w_k}$$

Let us fix such a compact K. It turns out that it suffices to show that for every $\varepsilon > 0$, if $\eta > 0$ is small enough,

$$\text{for } \gamma \in \{-1,1\}, \quad \begin{cases} \sum_{j\geq 1} \sup_{q\in K} j 2^{j\left(\tau(q)+\gamma\eta\tau'(q)-\varepsilon\eta\right)} \mathbb{E}\big(f_{j,\varepsilon,\eta}(q)\big) < \infty \\ \sum_{j\geq 1} \sup_{q\in K} 2^{j\left(\tau(q)+\gamma\eta\tau'(q)-\varepsilon\eta\right)} \mathbb{E}\big(|f'_{j,\varepsilon,\eta}(q)|\big) < \infty \end{cases} \tag{8.3}$$

Indeed, if (8.3) holds then, with probability one,

$$\sum_{j\geq 1} 2^{j(\tau(\inf(K))+\gamma\eta\tau'(\inf(K))-\varepsilon\eta)} f_{n,\varepsilon,\eta}(\inf K) < \infty,$$

$$\text{and} \qquad \int_K \sum_{j\geq 1} \left| \frac{d}{dq}\left(2^{j(\tau(q)+\gamma\eta\tau'(q)-\varepsilon\eta)} f_{j,\varepsilon,\eta}(q)\right)\right| dq < \infty.$$

Hence the series $\sum_{j\geq 1} 2^{j(\tau(q)+\gamma\eta\tau'(q)-\varepsilon\eta)} f_{j,\varepsilon,\eta}(q)$ converges uniformly on K (a similar approach was initially used to get the main result in [6]).

It follows from Lemma 6 in [6] that for η small enough and $\gamma \in \{-1, 1\}$, one has

$$C_K(\eta) = \sup_{q\in K,\ j\geq 1, v,w\in\mathcal{A}^j} \mathbb{E}\Big(\Big|\frac{d}{dq}(Y_1(v)^{-\gamma\eta} Y_q(w))\Big|\Big) + \mathbb{E}\left(Y_1(v)^{-\gamma\eta} Y_q(w)\right) < \infty$$

$$\text{and } \ C'_K(\eta) = \sup_{q\in K, j\geq 1, v,w\in\mathcal{A}^j} \frac{\mathbb{E}\Big(\Big|\frac{d}{dq} W_v^{-\gamma\eta} W_w^q\Big|\Big)}{\mathbb{E}\left(W_v^{-\gamma\eta} W_w^q\right)} < \infty.$$

Taking into account the fact that the W's are mutually independent, one gets

$$\mathbb{E}\left(|f'_{j,\varepsilon,\eta}(q)|\right) \leq C_K(\eta)(1 + jC'_K(\eta)) g_{j,\varepsilon,\eta}(q),$$

$$\text{where} \qquad g_{j,\varepsilon,\eta}(q) = \sum_{v,w\in\mathcal{A}^j :\, \delta(v,w)\leq 1} \ \prod_{k=1}^{j} \mathbb{E}\left(W_{v_1\cdots v_k}^{-\gamma\eta} W_{w_1\cdots w_k}^q\right).$$

One also has $\mathbb{E}\left(f_{j,\varepsilon,\eta}(q)\right) \leq C_K(\eta) g_{j,\varepsilon,\eta}(q)$.

A common way to represent the pairs of words (v, w) is the following. Let ρ_k be the word consisting of k consecutive zeros and let λ_k be the word consisting of k consecutive 1. A representation of the set of pairs (v, w) in $\mathcal{A}^j$ such that $i(w) = i(v) + 1$ is as follows:

$$\bigcup_{k=0}^{j-1} \ \bigcup_{u\in\mathcal{A}^{j-1-k}} \left\{(u.0.\lambda_k, u.1.\rho_k)\right\}. \tag{8.4}$$

Then, splitting $g_{j,\varepsilon,\eta}(q)$ into the sum over the identical pairs (w, w) and the sum over the pairs (v, w) such that $\delta(v, w) = 1$, and using (8.4), one obtains

$$g_{j,\varepsilon,\eta}(q) = 2^{-j\widetilde{\tau}(q+\gamma\eta)} + h_{j,\varepsilon,\eta}(q),$$

where $h_{j,\varepsilon,\eta}(q) = 2\sum_{k=0}^{j-1} 2^k \left(\mathbb{E}(W^{q+\gamma\eta})\right)^k \left(\mathbb{E}(W^{\gamma\eta})\mathbb{E}(W^{(q)})\right)^{j-k}$. One then sees that

$$\begin{aligned} h_{j,\varepsilon,\eta}(q) &= 2\left(\mathbb{E}(W^{\gamma q})\mathbb{E}(W^q)\right)^j \sum_{0\le k\le j-1}\left[\frac{2\mathbb{E}(W^{q+\gamma\eta})}{\mathbb{E}(W^{\gamma q})\mathbb{E}(W^q)}\right]^k \\ &\le 2^{1-j(2+\widetilde{\tau}(\gamma\eta)+\widetilde{\tau}(q))}\sum_{0\le k\le j-1} 2^{k(2-\widetilde{\tau}(q+\gamma\eta)+\widetilde{\tau}(\gamma\eta)+\widetilde{\tau}(q))} \\ &\le 2^{1-j(2+\widetilde{\tau}(\gamma\eta)+\widetilde{\tau}(q))}\sum_{0\le k\le j-1} 2^{k(2-\gamma\eta\widetilde{\tau}'(q)+\widetilde{\tau}(\gamma\eta)-\eta\varepsilon_q(\eta))} \\ &\le \frac{2^{1-j(\widetilde{\tau}(q)+\gamma\eta\widetilde{\tau}'(q)+\eta\varepsilon_q(\eta))}}{2^{(2-\gamma\eta\widetilde{\tau}'(q)+\widetilde{\tau}(\gamma\eta)-\eta\varepsilon_q(\eta))}-1}, \end{aligned}$$

with $\varepsilon_q(\eta)\to 0$ uniformly on K when $\eta\to 0$. Then, it is easily seen that (8.3) holds if η is small enough.

The second part of Theorem 5 comes from the fact that if $\alpha\notin\overline{\{\widetilde{\tau}'(q),\ q\in\mathcal{J}\}}$, then $\widetilde{\tau}^*(\alpha)<0$. Thus, using that $\tau=\widetilde{\tau}$ on $\mathcal{J}$, one has $\tau^*(\alpha)<0$.

8.2 Stable Lévy measures

In this section, we use the sets $\widetilde{E}^\mu_\alpha$ defined in (3.3) instead of the E^μ_α. Theorem 6 follows if the following property holds: With probability one, μ obeys the multifractal formalism at every $\alpha\in[0,1/\beta]$ with $\dim(\widetilde{E}^\mu_\alpha)=\beta\alpha$ and $\tau^*(\alpha)<0$, $\forall\alpha>1/\beta$. Let us denote by $\left(X_\beta(t)\right)_{t\in[0,1]}$ the stable subordinator such that, almost surely, $\mu([0,t])=X_\beta(t)$ for all $t\in[0,1]$.

We begin by estimating by above the Hausdorff dimension of the sets $\widetilde{E}^\mu_\alpha$. Let us estimate the function τ. One knows that $\mu(I_w)$ is a copy of $2^{-j/\beta}X(1)$ for every $j\ge 1$ and $w\in\mathcal{A}^j$. Moreover, $\mathbb{E}\big(X(1)^q\big)<\infty$ for every $q\in(-\infty,\beta)$. This yields that for every $j\ge 1$ and for every couple $(q,t)\in(-\infty,\beta)\times\mathbb{R}$, ($C_j(q,t)$ is defined in (3.2))

$$\mathbb{E}\big(C_j(q,t)\big)=2^{(1+t-q/\beta)j}\mathbb{E}\big(X(1)^q\big). \tag{8.5}$$

For every $q\in\mathbb{R}$, let us introduce the mapping

$$\widetilde{\tau}(q)=\begin{cases}-1+q/\beta & \text{if } q\in(-\infty,\beta)\\ 0 & \text{otherwise.}\end{cases}$$

It follows from (8.5) that for every $q\in]-\infty,\beta[$ and $\varepsilon>0$, with probability one $\sum_{j\ge1}C_j(q,\widetilde{\tau}(q)-\varepsilon)<\infty$. Consequently, $\forall q\in]-\infty,\beta[$, $\tau(q)\ge\widetilde{\tau}(q)$.

Using that τ and $\widetilde{\tau}$ are continuous, τ is non-decreasing, and $\widetilde{\tau}(\beta)=0$, one deduces that $\tau\ge\widetilde{\tau}$ on $\mathbb{R}$ almost surely. Thus, almost surely, $\tau^*(\alpha)\le\widetilde{\tau}^*(\alpha)=\alpha\beta$ for all $\alpha\in[0,1/\beta]$ and $\tau^*(\alpha)\le\widetilde{\tau}^*(\alpha)=-\infty$ for all $\alpha>1/\beta$.

Finally, by Proposition 1 applied to $\widetilde{E}^\mu_\alpha$, with probability one, $\dim(\widetilde{E}^\mu_\alpha)\le\beta\alpha$ for all $\alpha\in[0,1/\beta]$, and $\widetilde{E}^\mu_\alpha=\emptyset$ for all $\alpha>1/\beta$.

It remains to lower bound the dimensions. It is proved in [28] that, almost surely, the set E_α^X is empty if $\alpha > 1/\beta$ and $d_X(\alpha) = \beta\alpha$ if $\alpha \in [0, 1/\beta]$. Moreover, since a stable subordinator is not compensated (see [14]), if $\alpha \in [0, 1/\beta]$, the proof of Proposition 2 in [28] shows that for every $t \in E_\alpha^X$, $\forall\, \varepsilon > 0,\ \exists\, C > 0,\ \forall s \in [0,1] \setminus \{t\},\ |X(s) - X(t)| \le C|s-t|^{\alpha-\varepsilon}$.

Using the triangular inequality, this implies that, with probability one, for every $\alpha \in [0, 1/\beta]$ and $t \in E_\alpha^X$, $\min(\underline{\alpha}_\mu^-(t), \underline{\alpha}_\mu(t), \underline{\alpha}_\mu^+(t)) \ge \alpha$.

On the other hand, because of Proposition 1 of [28], with probability one, for every $\alpha \in [0, 1/\beta]$, if $t \in E_\alpha^X$, for every $\varepsilon > 0$ there exists $(j_n)_{n\ge 1}$, an increasing sequence of integers, and $(t_n)_{n\ge1}$, a sequence of jump points of X such that $X(t_n) - X(t_n^-) \ge 2^{-j_n-1}$ and $|t - t_n| \le 2^{-j_n/(\alpha+\varepsilon)}$. Let $J_n = [j_n/(\alpha+\varepsilon)]-1$. It is straightforward to see that if t is not a dyadic point one can assume without loss of generality that $t_n \in I_{J_n, k_{J_n,t}}$. This implies $\mu(I_{J_n,k_{J_n,t}}) \ge 2^{-3} 2^{-(\alpha+\varepsilon)J_n}$, $\forall\, n \ge 1$, and $\min(\underline{\alpha}_\mu^-(t), \underline{\alpha}_\mu(t), \underline{\alpha}_\mu^+(t)) \le \alpha + \varepsilon$. Since this holds for every $\varepsilon > 0$ and the set of dyadic points is at most countable, we deduce that

$$\text{with probability one, for all } \alpha \in (0, 1/\beta],\quad \dim(\widetilde{E}_\alpha^\mu) \ge \beta\alpha. \tag{8.6}$$

Let us finish with the case $\alpha = 0$. By construction, with probability one, the deterministic set D of dyadic points in $[0,1]$ does not contain any jump point of X. So $E_0^X \setminus D$, which contains the jump points of X, is not empty and (8.6) holds for $\alpha = 0$.

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