The singularity spectrum of Lévy processes in multifractal time

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1 Introduction

The interest for multifractal stochastic processes is mainly motivated by the need for accurate models in the study of the variability of wild signals. These locally irregular signals come from physical phenomena such as fully developed turbulence, TCP Internet traffic, variations of financial prices, or heart beats.

Fractional Brownian Motions (FBM), Lévy processes and multiplicative cascades are frequently used when modeling these phenomena. However, these processes are partly satisfactory for different reasons. FBM are monofractal, and thus have the same Hölder exponents at every point. Lévy processes are multifractal, but their multifractal spectrum takes a very specific linear increasing shape. Finally the multifractal multiplicative cascades only generate non-decreasing processes.

Other kinds of multifractal models were thus studied to go beyond these limitations. For instance, Gaussian processes with non-constant prescribed Hölder exponents are introduced in [2]. Another approach consists in generating multifractal random wavelet series [23,7].

A third point of view consists in performing a multifractal change of time in a given stochastic process $(X_t)_{t\geq 0}$. More precisely, given an atomless multifractal positive Borel measure μ on $\mathbb{R}_+$ supported by an interval of the form $[0,T]$

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($T \in (0,\infty)$), then the process $X \circ \mu([0,t])$ is considered. This process shall be viewed as the process X in (possibly multifractal) time μ.

The simplest situation lies in taking X equal to a monofractal process, like FBM (see [32,3,14] and Section 6). In this case, due to the monofractality property, the multifractal nature of $X \circ \mu$ follows almost straightforward from the one of μ (see Section 6). In the situation where it is assumed that X also has multifractal sample paths, the multifractal time change creates more interesting structures, both from the modeling and mathematical viewpoints (see for instance [37] for preliminary results on this topic, especially concerning large deviation spectra). The study of the sample paths multifractal properties is far more delicate than in the monofractal case. To our knowledge it has never been achieved in a non-trivial case.

This paper deals with the case where X is a Lévy process. We provide conditions on the measure μ under which the multifractal nature of the sample paths of the process $(Z_t = X \circ \mu([0,t]))_{t\geq 0}$ can be described. Before going further, let us detail the reason which led us to consider this problem.

Let b be an integer ≥ 2 and $W = (W_0, \dots, W_{b-1})$ a positive random vector. Then consider in the space of Laplace transforms of probability distributions ϕ on $\mathbb{R}_+$ the equation

$$\phi(u) = \mathbb{E}\left(\prod_{i=0}^{b-1} \phi(uW_i)\right), \quad \forall\, u \geq 0. \tag{1}$$

This equation, referred to as the smoothing transformation, is solved in [15,18]. It comes from the modeling of fully developed turbulence [31,30] and of interacting particles systems. Subsequently, the problem is then to find all the non-trivial solutions (i.e. $\not\equiv 1$) of (1). The mapping

$$\varphi_W : q \in \mathbb{R} \mapsto -\log_b \mathbb{E}\Big(\sum_{i=0}^{b-1} W_i^q\Big) \in \mathbb{R} \cup \{-\infty\}. \tag{2}$$

is implicated in the problem's resolution. Indeed, under the assumption that $\varphi_W(p) > -\infty$ for some $p > 1$, it is proved in [15] that (1) has non-trivial solutions if and only if there exists $\beta \in (0,1]$ such that $\varphi_W(\beta) = 0$ and $\varphi_W'(\beta) \geq 0$. As a consequence of the concavity of the mapping φ_W, such a β is unique and

$$\beta = \inf\{\beta' \in [0,1] : \varphi_W(\beta') = 0\}.$$

It is worth noting that the existence of non-trivial solutions in the general frame is almost entirely based on the existence of a non-trivial solution in the case $\beta = 1$ with $\varphi_W'(1) > 0$. Moreover, in this case, a fundamental non-trivial solution is given by the Laplace transform of the probability distribution of $\|\mu_W\|$, where μ_W is an independent multiplicative cascade on $[0,1]$ generated

by the random vector $W = (W_0, \dots, W_{b-1})$ used in (1), see [31,26] and Section 7 for the construction of μ_W. This type of multiplicative cascade measures has been extensively studied in [25,19,16,34,1,4]. Their well-known multifractal properties are closely related to φ_W and (1).
Therefore, as soon as $\varphi_W(1) = 0$ and $\varphi'_W(1) > 0$, it is possible to naturally associate the non-trivial stochastic process $(Z_{W,t})_{t\in[0,1]} = \big(\mu_W([0,t])\big)_{t\in[0,1]}$ with (1) such that the Laplace transform of $Z_{W,1}$ resolves (1). Moreover, this process $Z_{W,\cdot}$ is completely characterized by a statistical self-similarity property (see (39) in Section 7).

This raises the problem of finding a natural process satisfying the same properties in the general case $\beta \in (0,1]$. Let us recall how the solution ϕ of (1) in the case $\beta \in (0,1)$, $\varphi_W(\beta) = 0$ and $\varphi'_W(\beta) > 0$, is deduced from the construction of $\|\mu_W\|$ in [15,18]. First, the random vector $W_\beta = (W_0^\beta, \dots, W_{b-1}^\beta)$ is considered. By construction one gets that $\varphi_{W_\beta}(1) = 0$ and $\varphi'_{W_\beta}(1) > 0$, and we are back to the situation described above. Let ϕ_β be the Laplace transform of $\|\mu_{W_\beta}\|$. A non-trivial solution of (1) is then given by the mapping $\phi : u \mapsto \phi_\beta(u^\beta)$. Remark that this mapping is also the Laplace transform of the product $S_\beta \cdot \|\mu_{W_\beta}\|^{1/\beta}$, where S_β is a positive β-stable variable independent of $\|\mu_{W_\beta}\|$. Nevertheless, this observation does not provide a way to construct a stochastic process $(Z_{W,t})_{t\in[0,1]}$ associated with ϕ and fulfilling the statistical self-similarity property (39).

We obtain such a process as follows. Let X_β be a β-stable Lévy subordinator independent of μ_{W_β}. Consider the stochastic process

$$Z_{W,t} = X_\beta\Big(\mu_{W_\beta}([0,t])\Big) = X_\beta(Z_{W_\beta,t}) \quad (t \in [0,1]). \tag{3}$$

This process has the form of a Lévy process in multifractal time. This process $(Z_{W,t})_{t\in[0,1]}$ possesses the required properties. Indeed, one notices that

$$Z_{W,1} = X_\beta\Big(\|\mu_{W_\beta}\|\Big) \overset{d}{\equiv} S_\beta \cdot \|\mu_{W_\beta}\|^{1/\beta},$$

where $\overset{d}{\equiv}$ means equality in distribution. In addition, since X_β has by construction independent increments and is independent of μ_{W_β}, the increments of $Z_{W,t}$ also satisfy a statistical self-similarity property (see (39) in Section 7).

Equation (1) can also be considered in the space of characteristic functions of probability distributions on $\mathbb{R}$. It is shown in [29] that if there exists $\beta \in (1,2]$ such that $\varphi_W(\beta) = 0$ and $\varphi'_W(\beta) \geq 0$, (1) possesses a non-trivial non-positive solution. The solution when $\beta \in (0,1]$ is constructed in [15,18]. Actually, this solution can be viewed as the stochastic process $(Z_{W,t})_{t\geq 0}$ formally defined as in (3), but with a symmetric β-stable Lévy process X_β. Again, the multifractal nature of $(Z_{W,t})_{t\geq 0}$ appears to be related to φ_W.

Let us now resume the problem we address (i.e. to perform the multifractal analysis of a Lévy process in multifractal time) and our results.

First, the local regularity of a function f is measured in this paper as follows. Let $d \geq 1$, I a non-trivial subinterval of $\mathbb{R}_+$, and $f : I \to \mathbb{R}^d$. If $x \in I$, the *pointwise Hölder exponent* $h_f(x)$ of f at x is defined[1] by

$$h_f(x) = \liminf_{y \to x,\ y \neq x} \frac{\log |f(y) - f(x)|}{\log |y - x|}, \tag{4}$$

where $| \cdot |$ stands for the Euclidean norm, with the convention $|\log(0)| = \infty$.

Then the *multifractal nature* of f is expressed in terms of the size of the levels sets E_h^f of the function $h_f(\cdot)$ defined by $E_h^f = \{x \in I : h_f(x) = h\}$ ($h \geq 0$). This size is measured by the Hausdorff dimension (denoted dim, see Definition 3). Thus we focus on the estimation of the mapping

$$d_f : h \geq 0 \mapsto \dim E_h^f,$$

which is called *singularity spectrum* or *Hausdorff multifractal spectrum* of f.

The singularity spectrum of Lévy processes $(X_t)_{t \geq 0}$ – which corresponds in our context to the case where the measure μ equals the Lebesgue measure – is performed in [22] (see Theorem 1 below). There is no time change in this case: Lévy processes have with probability 1 a non-trivial linear multifractal spectrum. This typical shape is explained by the fact that the jump points of Lévy processes satisfy a *ubiquity* property with respect to the Lebesgue measure (the notion of ubiquity is detailed in Section 3.4).

In our context, when the measure μ is not monofractal, that is when the Hölder exponent function of the measure μ

$$h_\mu : t \mapsto \liminf_{r \to 0^+} \frac{\log \mu(B(t,r))}{\log(r)} \tag{5}$$

possesses several non-trivial level sets, the situation becomes subtler. We prove that the local behavior of the process $(Z_t = X \circ \mu([0,t]))_{t \geq 0}$ is closely related to some *conditioned ubiquity* properties (see Section 3.4), which combine conditions on the jump points of (Z_t) with conditions on the local behavior of μ.

[1] This definition is not the most common one. It may differ from the usual one, that we denote $H_f(x)$, which may involve a polynomial ([20]). If $h_f(x) \in \mathbb{R}^+ \setminus \mathbb{N}^*$, $h_f(x) = H_f(x)$. Nevertheless $h_f(x)$ is the natural notion here. Indeed, the study of (Z_t) requires information on the local behavior of $t \mapsto \mu([0,\cdot])$, that is on the Hölder exponents of the measure μ. These exponents are in general more tractable by using a definition similar to (4) than with the definition of [20].

Understanding these properties enables us to compute the singularity spectrum d_Z, under suitable assumptions. These technical assumptions are fulfilled by several classes of statistically self-similar measures μ with a construction based on ultiplicative cascade schemes, for instance some $\mathbb{R}_+$-martingales (like μ_W above) in the sense of [24,6] or random Gibbs measures (see [9,10]).

Before exposing our results, let us start by recalling precisely the theorem obtained in [22]. Let $X = (X_t)_{t\geq 0}$ be a $\mathbb{R}^d$-valued Lévy process. Recall that X has stationary independent increments and that its characteristic function takes the form $\mathbb{E}\Big(e^{i\langle\lambda|X_t\rangle}\Big) = e^{-t\psi(\lambda)}$, where

$$\psi(\lambda) = i\langle a|\lambda\rangle + Q(\lambda)/2 + \int_{\mathbb{R}^d}\Big(1 - e^{i\langle\lambda|x\rangle} + i\langle\lambda|x\rangle\mathbf{1}_{|x|\leq 1}\Big)\ \pi(dx)$$

and where $a\in\mathbb{R}^d$, Q is a quadratic form, and π is a Radon measure on $\mathbb{R}^d\setminus\{0\}$, called Lévy measure of X, satisfying

$$\int (1\wedge|x|^2)\pi(dx) < \infty. \tag{6}$$

Define the Blumenthal-Getoor exponent of X as

$$\beta = \inf\Big\{\gamma\geq 0 : \int_{|x|\leq 1}|x|^\gamma\pi(dx) < \infty\Big\}.$$

One always has $\beta\in[0,2]$. Remark that

$$\beta = \sup\Big(0, \limsup_{j\to+\infty} j^{-1}\log_2 C_j\Big), \text{ where } C_j = \int_{2^{-j-1}\leq|x|\leq 2^{-j}}\pi(dx)\ (j\geq 1). \tag{7}$$

We focus on the pointwise Hölder exponents of sample paths of X, thus without loss of generality we omit the jump points generated by the compound process with intensity $\mathbf{1}_{\{|x|>1\}}\,\pi(dx)$. When $\int(1\wedge|x|)\,\pi(dx) < \infty$, there are also several ways to write X as the sum of a Brownian motion B with drift $a'\in\mathbb{R}^d$ and covariance matrix Q and of a Lévy process $\widetilde{X}$ of Lévy measure $\mathbf{1}_{\{|x|\leq 1\}}\,\pi(dx)$, even when requiring that B and $\widetilde{X}$ are independent.

For $j\geq 0$, let $\pi_j(dx) = \mathbf{1}_{\{2^{-j-1}<|x|\leq 2^{-j}\}}\pi(dx)$, and let Y_j, $j\geq 0$, be independent compound Poisson processes of respective Lévy measure π_j.
We then choose $\widetilde{X}$ as follows:

$$\widetilde{X}_t = \sum_{j\geq 0} X_j(t) \text{ where } X_j(t) = \begin{cases} Y_j(t) & \text{if } \beta<1,\\ Y_j(t) - \int x\pi_j(dx) & \text{if } \beta\geq 1.\end{cases} \tag{8}$$

Then a general Lévy process (with jumps of norm ≤ 1) has the form

$$X = \widetilde{X} + B(a', Q), \tag{9}$$

where $B(a', Q)$ is a Brownian motion with drift $a' \in \mathbb{R}^d$ and covariance matrix Q, independent of $\widetilde{X}$ (of course if $Q = 0$ then B is degenerate).

Let us now state the theorem of [22] using the pointwise Hölder exponent introduced above in (4) instead of the classical one.
By convention, $\dim E = -\infty$ means that the set E is empty.

Theorem 1 *Let X be a Lévy process decomposed in the form $\widetilde{X} + B(a', Q)$ as in (8) and (9), and consider the associated process $\widetilde{X}$. Suppose that $\beta \in (0, 2]$ and $\sum_{j\geq 1} 2^{-j}\sqrt{C_j \log(1 + C_j)} < +\infty$ (this holds as soon as $\beta < 2$).*
With probability one, $d_{\widetilde{X}}(h) = \beta h$ if $h \in [0, 1/\beta]$ and $-\infty$ otherwise.

The influence of $B(a', Q)$ is also studied in [22], and the corresponding result is recalled in Theorem 3.

We now consider a positive Borel measure μ with a support equal to $[0, 1]$ and its integral F, i.e. F is the mapping $u \in [0, 1] \mapsto \mu([0, u])$. Let $(Z_u)_{u\in[0,1]}$ be the Lévy process in multifractal time F (or μ) given by $(Z_u = X_{F(u)})_{u\in[0,1]}$.

If μ is a typical multifractal measure, then F is a multifractal non-decreasing function. We also assume that μ is atomless, hence F is continuous on $[0, 1]$. We use the pointwise exponent of μ defined in (5). If $h \geq 0$, the level sets E^μ_h of the measure μ are defined as $E^\mu_h = \{u : \ h_\mu(u) = h\}$. Finally, the singularity spectrum (or Hausdorff multifractal spectrum) of μ is the mapping $d_\mu : h \mapsto \dim E^\mu_h$.

The so-called scaling function τ_μ associated with the measure μ is involved in our result. It is classically defined for positive Borel measures μ on $[0, 1]$ as

$$\tau_\mu : q \mapsto \liminf_{j\to+\infty} -j^{-1} \log_2 \sum_{0\leq k\leq 2^j-1} \mu\Big([k2^{-j}, (k + 1)2^{-j})\Big)^q. \tag{10}$$

The dyadic basis chosen in the definition (10) is not a restriction. Indeed, since $\mathrm{supp}\mu = [0, 1]$, another integer basis $b \geq 2$ would give the same value for τ_μ.

The Legendre transform f^* of a function $f : \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}$ is defined as $f^* : h \mapsto \inf_{q\in\mathbb{R}} hq - f(q)$.

Roughly speaking, our result yields the singularity spectrum d_Z of Z when the measure μ obeys the *multifractal formalism* in the sense that $d_\mu(h) = \tau^*_\mu(h)$ for all h (for detailed studies of multifractal formalisms for measures, the reader is referred to [13,35]). This property holds for many classes of statistically self-similar measures μ. These measures also satisfy three technical conditions **C1-3** invoked in our statement. For sake of shortness in this introduction, these conditions are specified in Section 3.4. The reader should keep in mind that these conditions are fulfilled by multinomial measures as well as by the

independent multiplicative cascade μ_W introduced above, under suitable conditions on W.
Among our assumptions in Section 3.5, we shall keep this property in mind:

$$\tau'_\mu(1) \text{ exists and is strictly positive.} \tag{11}$$

All these nice statistically self-similar measures have the common property that their lower and upper dimensions coincide and $\dim_*(\mu) = \dim^*(\mu) = \tau'_\mu(1) > 0$ (see [33] for the corresponding definitions).

We shall prove the following result, which includes Theorem 1 as the special case where μ is the Lebesgue measure.

Theorem 2 *Let X be a Lévy process decomposed in the form $\widetilde{X} + B(a', Q)$ as in (8) and (9). Suppose that $\beta \in (0,2]$, and $\sum_{j\geq 1} 2^{-j}\sqrt{C_j \log(1+C_j)} < +\infty$. Let μ be an atomless positive Borel measure whose support is $[0,1]$, such that (11) and* **C1** *hold.*
Let us introduce the exponents $h_{\mu,\beta} = \tau'_\mu(1)/\beta$ and $\alpha_{\max} = \sup\{\alpha : \tau^*_\mu(\alpha) \geq 0\}$.

Let $(\widetilde{Z}_u)_{u\in[0,1]}$ be the stochastic process defined by $\widetilde{Z}(u) = \widetilde{X}_{\mu([0,u])}$ (i.e. one does not take into account the influence of $B(a', Q)$ in the decomposition (9)).

With probability one, one has:

- *For every $h \in [0, h_{\mu,\beta})$, one has $d_{\widetilde{Z}}(h) \leq \beta h$.*
  Moreover, if **C2**$(h_{\mu,\beta})$ *holds, then for every $h \in [0, h_{\mu,\beta})$, $d_{\widetilde{Z}}(h) = \beta h$.*
- *If $h \in [h_{\mu,\beta}, \alpha_{\max}/\beta]$, $d_{\widetilde{Z}}(h) \leq \tau^*_\mu(\beta h)$.*
  Moreover, if **C3**(βh) *holds, then $d_{\widetilde{Z}}(h) = \tau^*_\mu(\beta h)$.*

The singularity spectrum of $\widetilde{Z}$ is thus composed of two parts (see Figure 1): First a linear part of slope $1/\beta$, then a concave part which is a dilated and translated version of (a part of) the singularity spectrum of the initial measure μ. The typical shape reflects the combination of an additive structure (the Lévy process) with a multiplicative structure (the multifractal measure μ).

Remark that the singularity spectrum of $\widetilde{Z}$ is obtained as the Legendre transform of the function

$$\tau_{\mu,\beta}(q) = \begin{cases} \tau_\mu(q/\beta) & \text{if } q \leq \beta, \\ 0 & \text{otherwise} \end{cases}$$

as soon as **C2**$(h_{\mu,\beta})$ and **C3**(h) for all $h \in [\tau'_\mu(1), \alpha_{\max})$ hold.

Once again, examples of measures illustrating our result are Gibbs measures and their random counterparts studied in [17,27,9], and of course the independent random cascades μ_W mentioned above in the study of the fixed points of

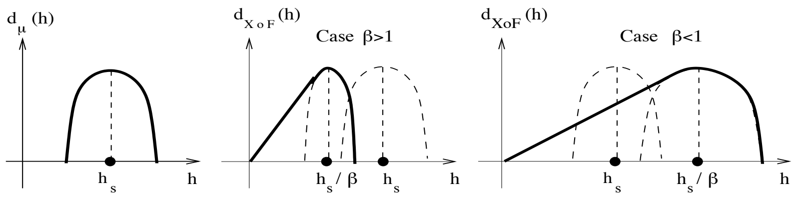

Fig. 1. Typical multifractal spectra of **Left:** a statistically self-similar measure μ, **Middle:** a Lévy process in multifractal time $\widetilde{X} \circ F$ when $\beta > 1$, and **Right:** when $\beta \leq 1$. Here h_s is the Lebesgue-almost sure exponent, i.e. $h_s = \tau_\mu'(0^+)$.

the smoothing transformation (1). Other examples are the compound Poisson cascades and other $\mathbb{R}_+$-martingales studied in [5,6,10].

Let us now treat the general case, i.e. the influence of the drift and of the Brownian component.

Theorem 3 *Under the assumptions of Theorem 2, let us introduce the exponents* $\widetilde{h}_{\mu,\beta} = \inf\{h \geq 0 : \beta h < \tau_\mu^*(h)\}$ *if* $\beta < 1$ *and* $\overline{h}_{\mu,\beta} = \inf\{h \geq 0 : \beta h < \tau_\mu^*(2h)\}$. *One always has* $\widetilde{h}_{\mu,\beta} < h_{\mu,\beta}$ *and* $\overline{h}_{\mu,\beta} \leq \tau_\mu'(1)/2 \leq h_{\mu,\beta}$.

Let us consider the two mappings ($\widetilde{D}_{\mu,\beta}$ *is defined if* $\beta < 1$*)*

$$\widetilde{D}_{\mu,\beta}(h) = \begin{cases} \beta h & \text{if } h \in [0, \widetilde{h}_{\mu,\beta}) \\ \tau_\mu^*(h) & \text{if } h \in [\widetilde{h}_{\mu,\beta}, \alpha_{\max}] \\ -\infty & \text{otherwise} \end{cases} \quad \text{and} \quad \overline{D}_{\mu,\beta}(h) = \begin{cases} \beta h & \text{if } h \in [0, \overline{h}_{\mu,\beta}) \\ \tau_\mu^*(2h) & \text{if } h \in [\overline{h}_{\mu,\beta}, \frac{\alpha_{\max}}{2}] \\ -\infty & \text{otherwise.} \end{cases}$$

Let $(Z_u)_{u \in [0,1]}$ *be the stochastic process defined by* $Z(u) = X_{\mu([0,u])}$.

1. *Suppose that* $Q = 0$ *and (*$a' = 0$ *if* $\beta < 1$*). With probability one, the same conclusions as for Theorem 2 occur here.*

2. *Suppose that* $Q = 0$, $\beta < 1$ *and* $a' \neq 0$. *With probability one,*

- $d_Z \leq \widetilde{D}_{\mu,\beta}$.

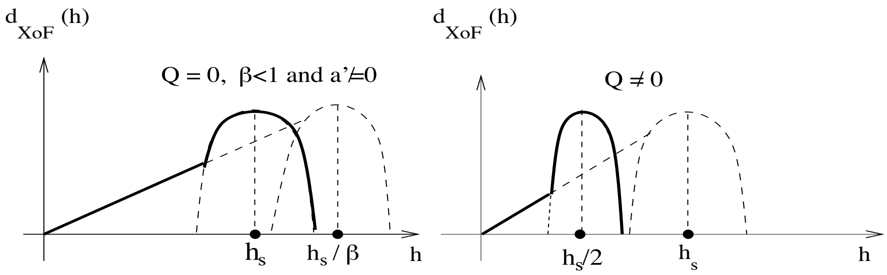

Fig. 2. Typical multifractal spectra of **Left:** a Lévy process in multifractal time $\widetilde{X} \circ F$ when $\beta > 1$, and **Right:** when $\beta \leq 1$.

• *If* $\mathbf{C2(}h_{\mu,\beta}\mathbf{)}$ *holds, for every* $h \in [0, \widetilde{h}_{\mu,\beta})$ *one has* $d_Z(h) = \widetilde{D}_{\mu,\beta}(h)$.

• *If* $\tau_\mu^*(\widetilde{h}_{\mu,\beta}) = \beta\widetilde{h}_{\mu,\beta}$ *and* $\mathbf{C2(}h_{\mu,\beta}\mathbf{)}$ *holds, or if* $\tau_\mu^*(\widetilde{h}_{\mu,\beta}) > \beta\widetilde{h}_{\mu,\beta}$ *and* $\mathbf{C3(}\widetilde{h}_{\mu,\beta}\mathbf{)}$ *holds, then* $d_Z(\widetilde{h}_{\mu,\beta}) = \widetilde{D}_{\mu,\beta}(\widetilde{h}_{\mu,\beta})$.

• *If* $h \in (\widetilde{h}_{\mu,\beta}, \alpha_{\max}]$ *and* $\mathbf{C3(}h\mathbf{)}$ *holds, then* $d_Z(h) = \widetilde{D}_{\mu,\beta}(h)$.

3. *Suppose that* $Q \neq 0$. *With probability one,*

• $d_Z \leq \overline{D}_{\mu,\beta}$.

• *If* $\mathbf{C2(}h_{\mu,\beta}\mathbf{)}$ *holds, for every* $h \in [0, \overline{h}_{\mu,\beta})$ *one has* $d_Z(h) = \overline{D}_{\mu,\beta}(h)$.

• *If* $\tau_\mu^*(2\overline{h}_{\mu,\beta}) = \beta\overline{h}_{\mu,\beta}$ *and* $\mathbf{C2(}h_{\mu,\beta}\mathbf{)}$ *holds, or if* $\tau_\mu^*(2\overline{h}_{\mu,\beta}) > \beta\overline{h}_{\mu,\beta}$ *and* $\mathbf{C3(}2\overline{h}_{\mu,\beta}\mathbf{)}$ *holds, then* $d_Z(\overline{h}_{\mu,\beta}) = \overline{D}_{\mu,\beta}(\overline{h}_{\mu,\beta})$.

• *If* $h \in (\overline{h}_{\mu,\beta}, \alpha_{\max}/2]$ *and* $\mathbf{C3(}2h\mathbf{)}$ *holds, then* $d_Z(h) = \overline{D}_{\mu,\beta}(h)$.

The conclusions of items **2.** and **3.** are simple consequences of the fact that respectively a linear drift and a Brownian component are added to the "pure" Lévy process $\widetilde{X}$. The corresponding spectra are simply obtained as supremum of two spectra. This explains their non-concave shapes (see Figure 2).

The paper is organized as follows.
Section 2 recalls some useful properties of measures.
Section 3 introduces the main tools used in the proof of Theorem 2. Properties of Poisson point processes are discussed, and estimates for the increments of $\widetilde{X}$ obtained in [22] are recalled. Then, results on heterogeneous ubiquitous systems (introduced in [11]) are stated, and conditions **C1-3** are defined.
Section 4 is devoted to the proof of Theorem 2 when $B(a', Q) \equiv 0$. Sections 5

and 6 complete the proof to yield the general case $B(a', Q) \not\equiv 0$.
Section 7 deals with the validity of condition $\mathbf{C2}(h_{\mu,\beta})$ for independent multiplicative cascades, which play a central role in the fundamental example (3).

2 Local regularity of measures

For every $j \geq 1$ and $k \in [0, \ldots, 2^j - 1]$, $I_{j,k} = [k2^{-j}, (k+1)2^{-j})$. $I_{j,k}^+$ and $I_{j,k}^-$ denote the intervals $I_{j,k} + 2^{-j}$ and $I_{j,k} - 2^{-j}$.

If $u \in (0,1)$, $\forall j \geq 1$, $I_j(u)$ denotes the unique dyadic interval of length 2^{-j} containing u. Then define $I_j^+(u) = I_j(u) + 2^{-j}$ and $I_j^-(u) = I_j(u) - 2^{-j}$.

The diameter of a set B is denoted by $|B|$. For the rest of the paper, the convention $\log(0) = -\infty$ is adopted.

Definition 1 *Let μ be a positive Borel measure on $[0,1]$. For $u_0 \in (0,1)$, the lower and upper Hölder exponents of μ at u_0 are respectively defined by*

$$\underline{\alpha}_\mu(u_0) = \liminf_{j\to+\infty} \frac{\log \mu(I_j(u_0))}{\log |I_j(u_0)|} \quad \textit{and} \quad \overline{\alpha}_\mu(u_0) = \limsup_{j\to+\infty} \frac{\log \mu(I_j(u_0))}{\log |I_j(u_0)|}$$

When $\underline{\alpha}_\mu(u_0) = \overline{\alpha}_\mu(u_0)$, their common value is denoted $\alpha_\mu(u_0)$ and called the Hölder exponent of μ at u_0.

The left and right lower and upper Hölder exponents of μ at u_0 are defined by

$$\begin{aligned} \underline{\alpha}_\mu^-(u_0) &= \liminf_{j\to+\infty} \frac{\log \mu(I_j^-(u_0))}{\log |I_j^-(u_0)|} \; \textit{and} \underline{\alpha}_\mu^+(u_0) = \liminf_{j\to+\infty} \frac{\log \mu(I_j^+(u_0))}{\log |I_j^+(u_0)|} \\ \textit{and} \quad \overline{\alpha}_\mu^-(u_0) &= \limsup_{j\to+\infty} \frac{\log \mu(I_j^-(u_0))}{\log |I_j^-(u_0)|} \; \textit{and} \overline{\alpha}_\mu^+(u_0) = \limsup_{j\to+\infty} \frac{\log \mu(I_j^+(u_0))}{\log |I_j^+(u_0)|}. \end{aligned}$$

Similarly, when they coincide, $\alpha_\mu^-(u_0)$ and $\alpha_\mu^+(u_0)$ denote their common value. Finally, one defines

$$\overline{h}_\mu(u_0) = \max(\overline{\alpha}_\mu^-(u_0), \overline{\alpha}_\mu(u_0), \overline{\alpha}_\mu^+(u_0))$$

and for $h \geq 0$

$$\overline{E}_h^\mu = \{u \in [0,1] : \overline{h}_\mu(u) = h\}.$$

One sees that (the exponent $h_\mu(\cdot)$ and its level sets E_h^μ are defined in (4))

$$h_\mu(u_0) = \min(\underline{\alpha}_\mu^-(u_0), \underline{\alpha}_\mu(u_0), \underline{\alpha}_\mu^+(u_0)) = \liminf_{r\to 0^+} \frac{\log \mu(B(u_0, r))}{\log |B(u_0, r)|}.$$

Definition 2 *If μ is a positive Borel measure on $[0,1]$, and $\alpha \geq 0$, one denotes by $\widetilde{E}^\mu_\alpha$ the set $\{x : \alpha^-_\mu(x) = \alpha_\mu(x) = \alpha^+_\mu(x) = \alpha\}$.*

The following proposition put together some classical results derived from the multifractal formalism for measures (see [13,35]). It provides upper bounds for the Hausdorff dimension of union of level sets of the Hölder exponents of a measure μ supported by $[0,1]$. The singularity spectrum d_μ and the scaling function τ_μ were introduced in Section 1.

Proposition 1 *Let μ be a positive Borel measure on $[0,1]$ and let $\alpha \geq 0$.*

(1) One has $\dim \widetilde{E}^\mu_\alpha \leq d_\mu(\alpha) \leq \tau^*_\mu(\alpha)$.
(2) If $\alpha \in [0, \tau'_\mu(0^+)]$ then $\dim \bigcup_{\alpha' \leq \alpha} E^\mu_{\alpha'} \leq \tau^*_\mu(\alpha)$.
(3) If $\alpha \geq \tau'_\mu(0^+)$ then $\dim \bigcup_{\alpha' \geq \alpha} \left(E^\mu_{\alpha'} \cup \overline{E}^\mu_{\alpha'}\right) \leq \tau^*_\mu(\alpha)$.
(4) If $\tau^*_\mu(\alpha) < 0$ then $E^\mu_\alpha = \emptyset$.

Next proposition follows from the definition of τ_μ and some Tchernov inequalities.

Proposition 2 *Let μ be a positive Borel measure on $[0,1]$. For every $\alpha \geq 0$, $C > 0$ and $\varepsilon > 0$, there exists a scale J such that $j \geq J$ implies*

$$\frac{\log\Big(\# \Big\{k \in \{0,..,2^j-1\} : \mu(I_{j,k}) \geq C 2^{-j(\alpha+\varepsilon)}\Big\}\Big)}{\log 2^j} \leq \sup_{\alpha' \leq \alpha+\varepsilon} \tau^*_\mu(\alpha') + \varepsilon.$$

Definition 3 *Let $s \geq 0$. The s-dimensional Hausdorff measure of a set E, $\mathcal{H}^s(E)$, is defined as*

$$\mathcal{H}^s(E) = \lim_{r \searrow 0} \mathcal{H}^s_r(E), \quad \text{with } \mathcal{H}^s_r(E) = \inf\Big\{\sum_i |E_i|^s\Big\},$$

the infimum being taken over all the countable families of sets E_i such that $|E_i| \leq r$ and $E \subset \bigcup_i E_i$. Then, the Hausdorff dimension of E, $\dim E$, is defined as $\dim E = \inf\{s \geq 0 : \mathcal{H}^s(E) = 0\}$.

3 Tools

In this section, we are given the Lévy process X, decomposed into the sum $X = \widetilde{X} + B(a', Q)$ described in (9).

3.1 Some notations

Let us denote by S the Poisson point process with intensity $\ell \otimes \pi$ associated with the Lévy process $X(t)$, where ℓ stands for the Lebesgue measure on $\mathbb{R}_+$ and π is the Lévy measure.

For every $j \geq 1$, let

$$G_j = \{t : (t,\lambda) \in S \text{ for some } \lambda \text{ such that } |\lambda| \in (2^{-j-1}, 2^{-j}]\}.$$

For $t \in G_j$, λ_t is the unique element $\lambda \in \mathbb{R}^d$ such that $(t,\lambda) \in S$. The jumps of the process $X_j(t)$ are thus exactly located at the points of G_j, and the value of the jump of X_j at $t \in G_j$ is λ_t.

For every $j \geq 1$ and for every $\delta > 0$, A_δ^j is the union of intervals

$$A_\delta^j = \bigcup_{t \in G_j} B(t, 2^{-j\delta}).$$

One clearly has $\bigcup_{t \in G_j} B(t, |\lambda_t|^\delta) \subset A_\delta^j$. Eventually, for every sequence $\widetilde{\delta} = \{\delta_j\}_j$ of non-negative numbers, let us denote

$$A_{\widetilde{\delta}} = \limsup_{j \to +\infty} A_{\delta_j}^j = \bigcap_{J \geq 1} \bigcup_{j \geq J} A_{\delta_j}^j. \tag{12}$$

3.2 Local regularity of the Lévy process $\widetilde{X}$

As a consequence of the work achieved by Jaffard in [22], one has the following properties of the increments of $\widetilde{X}$.

Proposition 3 *Let $\varepsilon > 0$. With probability one:*

Let $t_0 \geq 0$ be not a jump point of $\widetilde{X}(t)$, and let us write $h_X(t_0) = 1/\delta$ for some $\delta \geq \beta$. For η small enough, there exists $\varepsilon' > 0$ such that

$$\forall\, t \geq 0,\ |t - t_0| \leq \eta, \sum_{j \geq \frac{\log_2 |t-t_0|^{-1}}{\beta + \varepsilon'}} |X_j(t) - X_j(t_0)| \leq |t - t_0|^{1/(\beta+\varepsilon)} \tag{13}$$

$$\text{and } \ |X(t) - X(t_0)| \leq |t - t_0|^{1/(\delta+\varepsilon)}. \tag{14}$$

Moreover, still for $|t - t_0| \leq \eta$, if $\sum_{j < \frac{\log_2 |t-t_0|^{-1}}{\beta+\varepsilon'}} X_j(\cdot)$ has no jump point between

t and t_0, one has

$$\sum_{j<\frac{\log_2 |t-t_0|^{-1}}{\beta+\varepsilon'}} |X_j(t) - X_j(t_0)| \leq |t-t_0|^{1/(\beta+\varepsilon)}. \tag{15}$$

Equation (15) implies that the contribution of the sum of all the drifts associated with the processes $X_j(t)$, $j < \frac{\log_2 |t-t_0|^{-1}}{\beta+\varepsilon'}$, on a given interval $[t_0, t]$, is always less than $|t-t_0|^{1/(\beta+\varepsilon)}$.

3.3 Coverings and weak redundancy properties associated with Poisson point processes

It is known [38,22] that with probability 1, for every $\delta < \beta$, if the sequence $\widetilde{\delta}$ is constant equal to δ, then $A_{\widetilde{\delta}} = \mathbb{R}^+$ (recall (12)). An easy adaptation of the proof of Lemma 3 in [22] yields the following slightly stronger result.

Lemma 1 *With probability 1, there exists a non-decreasing non-negative sequence $\widetilde{\beta} = (\beta_j)_{j\geq 1}$ converging to β such that $A_{\widetilde{\beta}} = \mathbb{R}^+$.*

Notice that if the Lévy process is stable and if one can write in polar coordinates $\pi(dr, d\theta) = \alpha r^{-(1+\beta)}\, dr \nu(d\theta)$ with $\alpha \geq 1/2$ and ν a probability measure on the unit sphere, then the constant sequence $(\beta_j = \beta)_j$ can be chosen in the previous statement.

The problem of Poisson intervals covering is connected with the problem of counting the number of points of S which projection on $\mathbb{R}_+$ falls in a given dyadic interval $I_{j,k} = [k2^{-j}, (k+1)2^{-j})$. Next Lemmas 2 and 4 are devoted to this question.

Lemma 2 *For $\delta > \beta$ and $\widetilde{\varepsilon} = \{\varepsilon_j\}_{j\geq 1}$ a sequence of positive numbers, and for every integers j and k, let*

$$K_{j,k}^{\delta,\widetilde{\varepsilon}} = \#\Big\{t \in I_{j,k} : t \in G_{j'} \text{ for some } j' \in [j/\delta, \ldots, j/(\beta+\varepsilon_j)]\Big\}. \tag{16}$$

With probability 1, there exist two sequences $\{\varepsilon_j\}_{j\geq 1}$ and $\{\eta_j\}_{j\geq 1}$ of positive real numbers converging to 0 such that for every $\delta > \beta$, for every $j \geq 1$ large enough (depending on δ), for every $k \in \{0, \ldots, 2^j - 1\}$, one has $K_{j,k}^{\delta,\widetilde{\varepsilon}} \leq 2^{j\eta_j}$.

PROOF. By definition of β, there exists a positive non-increasing sequence $\{\varepsilon_j^1\}_j$ converging to zero such that $C_j \leq 2^{j(\beta+\varepsilon_j^1)}$.

Let $\delta > \beta$. For every $j \geq 1$ and $k \in \{0, \dots, 2^j - 1\}$, the random variable $K^\delta_{j,k}$ is a Poisson variable with intensity $C^\delta_j = 2^{-j} \sum_{j/\delta \leq j' \leq j/(\beta+\varepsilon^1_j)} C_{j'} \leq 2^{-j} \sum_{j/\delta \leq j' \leq j/(\beta+\varepsilon^1_j)} 2^{j'(\beta+\varepsilon^1_{j'})} \leq M_{j,\delta} 2^{j((\beta+\varepsilon^1_{[j/\delta]})/(\beta+\varepsilon^1_j)-1)}$, where $M_{j,\delta}$ is a constant equal to $2\frac{2^{(\beta+\varepsilon^1_{[j/\delta]})/(\beta+\varepsilon^1_j)}}{2^{\beta+\varepsilon^1_{[j/\delta]}}-1}$. In fact it is easily checked that the sequence $M_{j,\delta}$ can be bounded by a constant M_β independent of δ and j since the sequence $\{\varepsilon^1_j\}$ is bounded. Thus $C^\delta_j \leq M_\beta 2^{j((\beta+\varepsilon^1_{[j/\delta]})/(\beta+\varepsilon^1_j)-1)}$, for every j and every δ.

Moreover, since the sequence $\{\varepsilon^1_j\}$ is non-increasing and converges to zero as $j \to +\infty$, $(\beta + \varepsilon^1_{[j/\delta]})/(\beta + \varepsilon^1_j) - 1$ is bounded by $\{\varepsilon^{(\delta)}_j\} = \{2\varepsilon^1_{[j/\delta]}/\beta\}$, which is a non-increasing sequence depending on δ. Thus C^δ_j is bounded by $M_\beta 2^{j\varepsilon^{(\delta)}_j}$.

Let us consider $\varepsilon_j = \varepsilon^1_{[j/\log j]}$, $\forall j \geq 1$. For every $\delta > \beta$, for j large enough, one has $\varepsilon_j \geq \varepsilon^{(\delta)}_j$, and thus $C^\delta_j \leq 2^{j\varepsilon_j}$ (actually, without loss of generality, one can change a little bit the sequence $\{\varepsilon^{(\delta)}_j\}$ so that it takes into account the constant M_β).

Let us now recall a technical lemma proved in [22].

Lemma 3 *There exists $C' > 0$ and $D > 0$ such that if N is a Poisson variable of intensity $C \geq C'$, $\mathbb{P}(|N - C| \geq C/2) \leq e^{-DC \log C}$.*

By Lemma 3, for j large enough, if $C^\delta_j \geq j$, one gets for any integer k that $\mathbb{P}(K^{\delta,\widetilde{\varepsilon}}_{j,k} \geq \frac{3}{2} C^\delta_j) \leq 2^{-3j}$. Moreover, if $C^\delta_j \leq j$, using Stirling's formula one gets $\mathbb{P}(K^{\delta,\widetilde{\varepsilon}}_{j,k} \geq 2j) \leq 2^{-3j}$, still for j large enough. As a consequence, $\mathbb{P}\Big(K^{\delta,\widetilde{\varepsilon}}_{j,k} \geq 2j + 2C^\delta_j\Big) \leq 2^{-2j}$.

Let $P^\delta_j = \mathbb{P}\Big(\exists k \in \{0, \dots, 2^j - 1\} : K^{\delta,\widetilde{\varepsilon}}_{j,k} \geq 2j + 2C^\delta_j\Big)$. Using the above arguments, one gets $P^\delta_j \leq 2^{-2j} 2^j$, and $\sum_{j \geq 1} P^\delta_j < +\infty$. The Borel-Cantelli lemma implies that for every j large enough, for every $k \in \{0, \dots, 2^j - 1\}$, $K^{\delta,\widetilde{\varepsilon}}_{j,k} \leq 2j + 2C^\delta_j \leq 2^{j\eta_j}$, where $\eta_j = \varepsilon_j + 2(\log_2 j)/j$. This yields with probability one the uniform control over k of $K^{\delta,\widetilde{\varepsilon}}_{j,k}$ for every $\delta > \beta$, and then with probability one for all $\delta > \beta$ since the random functions $\delta \mapsto K^{\delta,\widetilde{\varepsilon}}_{j,k}$ are non-decreasing.

We need to introduce the notion of *weakly redundant system* in $\mathbb{R}_+$. This notion is later determinant to get upper bounds for the level sets of Hölder exponents.

Definition 4 *Let $(x_n)_{n \geq 0} \in \mathbb{R}^{\mathbb{N}}_+$ and $(\lambda_n)_{n \geq 0}$ a positive sequence converging* *to 0. For every $T > 0$ and $j \geq 0$, let us introduce the sets of indices*

$$\mathcal{T}_j = \left\{ n : x_n \in [0,T],\ 2^{-(j+1)} < \lambda_n \leq 2^{-j} \right\}. \tag{17}$$

The family $\{(x_n, \lambda_n)\}_{n\in\mathbb{N}}$ is said to form a weakly redundant system if for every $T > 0$ there exists a sequence of integers $(N_{T,j})_{j\geq 0}$ such that

(i) $\lim_{j\to\infty}(\log_2 N_{T,j})/j = 0$.

(ii) for every $j \geq 1$, $\mathcal{T}_j$ can be decomposed into $N_{T,j}$ pairwise disjoint subsets (denoted $\mathcal{T}_{j,1}, \ldots, \mathcal{T}_{j,N_{T,j}}$) such that for each $1 \leq i \leq N_{T,j}$, the family $\left\{ B(x_n, \lambda_n) :\ n \in \mathcal{T}_{j,i} \right\}$ is composed of disjoint balls.

Lemma 4 *Consider the Poisson point process $S = \bigcup_{j\geq 0} G_j$. Let $(\beta_j)_{j\geq 0}$ be a non-decreasing sequence converging to β.*

With probability one, the family $\bigcup_{j\geq 0}\{(t, |\lambda_t|^{\beta_j}) : t \in G_j\}$ forms a weakly redundant system.

PROOF. This is a direct consequence of the estimates obtained in the proofs of Lemmas 5 and 8 of [22] for the numbers $N_{j,k} = \#\{t \in G_j :\ t \in [k2^{-j}, (k+1)]2^{-j}\}$ when $\beta = 1$.

3.4 Heterogeneous ubiquity and Hausdorff dimensions of Cantor sets

General results of what we call "heterogeneous ubiquity" are obtained in [11] (see also [12]). Here, a simpler version adapted to our context is stated. It plays a similar role as the geometric Theorem 2 used in [22], but makes it possible to work out problems raised here by considering a multifractal time change. Some additional notations have to be introduced.

Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of points in $[0,1]$ and $\{l_n\}_{n\in\mathbb{N}}$ a sequence of positive real numbers converging to zero. Let $\delta > 1$. For every $n \in \mathbb{N}$ we set

$$I_n = [u_n - l_n, u_n + l_n],\ I_n^\delta = [u_n - l_n^\delta, u_n + l_n^\delta].$$

In addition, given an integer $b \geq 2$, for $u \in [0,1]$ one defines

$$\mathcal{B}_j(u) = \left\{ I_n : u \in I_n,\ l_n \in (b^{-(j+1)}, b^{-j}] \right\}, \tag{18}$$

$$\mathcal{B}_j^\delta(u) = \left\{ I_{j',k'} : \exists I_n \in \mathcal{B}_j(u) \text{ such that } I_{j',k'} \subset I_n^\delta \right\}. \tag{19}$$

Definition 5 *Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of points in $[0,1]$, and let $\{l_n\}_{n\in\mathbb{N}}$ be a sequence of positive real numbers converging to zero.*

Let μ be a positive Borel measure such that supp$(\mu) = [0,1]$ and (11) holds.

The system $\{(u_n, l_n)\}_n$ is said to form an heterogeneous ubiquitous system with respect to $(\mu, \tau'_\mu(1))$ if the following properties hold.

• **(1)** *There exists a non-increasing sequence $(\varphi_j)_{j\geq 0}$ with the following properties:*
- $\lim_{j\to\infty} \varphi_j = 0$, *$(j\varphi_j)_{j\geq 0}$ is non-decreasing at $+\infty$ and* $\lim_{j\to\infty} j\varphi_j = +\infty$.
- $\forall\, \varepsilon > 0$, *$(j(\varepsilon - \varphi_j))_{j\geq 0}$ is non-decreasing at $+\infty$,*
- *Properties* **(2), (3)** *and* **(4)** *below hold.*

• **(2)** *There exist an integer $b \geq 2$ such that*

- (2a) *μ-almost every $t \in [0,1]$ belongs to $\bigcap_{N\geq 0} \bigcup_{n\geq N} [u_n - l_n/2, u_n + l_n/2]$.*

- (2b) *For μ-almost every $t \in [0,1]$, there exists an integer $j(t)$ such that $\forall j \geq j(t)$, $\forall$ k such that $|k - k^b_{j,t}| \leq 1$,*

$$b^{-j(\tau'_\mu(1)+\varphi_j)} \leq \mu([kb^{-j}, (k+1)b^{-j}]) \leq b^{-j(\tau'_\mu(1)-\varphi_j)},$$

where $k^b_{j,t}$ is the unique integer k such that $t \in [kb^{-j}, (k+1)b^{-j})$. Thus **(2b)** *implies for μ-a.e. $t \in [0,1]$ a precise control of the μ-mass of the three b-adic intervals around t.*

• **(3)** *(Self-similarity of μ) For every b-adic subinterval L of $[0,1]$, let f_L denote the canonical affine mapping from L onto $[0,1]$. There exists a measure μ^L on L, equivalent to the restriction of μ to L, such that property* **(2b)** *holds for the measure $\mu^L \circ f_L^{-1}$ instead of the measure μ.*

Let $j_L = \log_b \left(|L|^{-1}\right)$ and for every $n \geq 1$, let

$$U_n^L = \left\{ t \in L : \left\{ \begin{array}{c} \forall\ j \geq n + j_L,\ \forall\ k,\ |k - k^b_{j,t}| \leq 1, \\ \mu^L\Big([kb^{-j}, (k+1)b^{-j}]\Big) \leq \left(\frac{b^{-j}}{|L|}\right)^{\tau'_\mu(1) - \varphi_{j-j_L}} \end{array} \right. \right\}.$$

The sets U_n^L clearly form a non-decreasing sequence in $[0,1]$, and by **(2b)** *and property* **(3)**, *$\bigcup_{n\geq 1} U_n^L$ is of full μ^L-measure. Then let us define*

$$n_L = \inf\Big\{ n \geq 1 :\ \mu^L(U_n^L) \geq \|\mu^L\|/2 \Big\}.$$

• **(4)** *(Control of the growth speed n_L and of the mass $\|\mu^L\|$)*

There exists a dense subset $\mathcal{D}$ of $(1,\infty)$ such that for every $\delta\in\mathcal{D}$, for μ-almost every $u\in[0,1]$, there exists an increasing sequence of integers $(j_k(u))_{k\geq 1}$ such that for every $k\geq 1$ there exists $L_k\in B^{\delta}_{j_k(u)}(u)$ satisfying $\lim_{k\to\infty}\frac{j_{L_k}}{j_k(u)}=\delta$ and

$$n_{L_k}\leq j_{L_k}\cdot\varphi_{j_{L_k}} \quad \text{and} \quad |L_k|^{\varphi_{j_{L_k}}}\leq\|\mu^{L_k}\|. \tag{20}$$

The next result is established in [11].

Theorem 4 *Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of points in $[0,1]$, let $\{l_n\}_{n\in\mathbb{N}}$ be a sequence of positive real numbers converging to zero. Let μ be a positive Borel measure such that supp$(\mu)=[0,1]$ and (11) holds.*

For every positive sequences $\widetilde{\varepsilon}=(\varepsilon_n)_{n\in\mathbb{N}}$ and $\widetilde{\delta}=(\delta_n)_{n\in\mathbb{N}}$, define the limsup set

$$S_\mu(\widetilde{\delta},\tau'_\mu(1),\widetilde{\varepsilon})=\bigcap_{N\geq 0}\ \bigcup_{n\geq N:\, l_n^{\tau'_\mu(1)+\varepsilon_n}\leq\mu(I_n)\leq l_n^{\tau'_\mu(1)-\varepsilon_n}} I_n^{\delta_n}.$$

Suppose that $\{(u_n,l_n)\}_n$ forms an heterogeneous ubiquitous system with respect to $(\mu,\tau'_\mu(1))$.

There exists a positive sequence $\widetilde{\varepsilon}$ converging to 0 such that for every $\delta\geq 1$, there exists a non-decreasing sequence $\widetilde{\delta}$ converging to δ as well as a positive Borel measure m_δ such that:
- $m_\delta(E)=0$ for every Borel set E such that $\dim E<\tau'_\mu(1)/\delta$,
- $m_\delta\Big(S_\mu(\widetilde{\delta},\tau'_\mu(1),\widetilde{\varepsilon})\Big)>0$.

In particular, $\dim\, S_\mu(\widetilde{\delta},\tau'_\mu(1),\widetilde{\varepsilon})\geq\tau'_\mu(1)/\delta$.

Moreover, if the system $\{(u_n,l_n)\}_{n\in\mathbb{N}}$ is weakly redundant (see Definition 4), one precisely has $\dim\, S_\mu(\widetilde{\delta},\tau'_\mu(1),\widetilde{\varepsilon})=\tau'_\mu(1)/\delta$.

The set $S_\mu(\widetilde{\delta},\tau'_\mu(1),\widetilde{\varepsilon})$ is constituted by points which are well approximated at rate $\delta>1$ by some points u_n, these points being selected according to the behavior of μ around u_n. Thus Theorem 4 emphasizes a ubiquity property *conditioned* by a measure μ, and shows the existence of exceptional points related simultaneously to the local behavior of the measure μ and to the approximation rate by the system $\{(u_n,l_n)\}_n$.

Remark 1 *For some classes of measures μ, it turns out that property* **(4)** *can be simplified in the stronger one: There exists $j_0\geq 0$ such that (20) holds for all b-adic interval L of generation larger than j_0. This is the case for* *instance for the class of random Gibbs measures described in [9]. Unfortunately independent random cascades do not satisfy this uniform property, and their study imposed the weaker condition* **(4)** *(see Sections 1 and 7 as well as [10]).*

3.5 Conditions **C1-3**

Let μ be an atomless positive Borel measure with a support equal to $[0,1]$.

Condition **C1**

There exist two positive constants γ_1 and γ_2 such that for every small enough sub-interval I of $[0,1]$, $|I|^{\gamma_1} \le \mu(I) \le |I|^{\gamma_2}$.

Condition $\mathbf{C2}(h_{\mu,\beta})$

Recall that $h_{\mu,\beta} = \tau'_\mu(1)/\beta$. By assumption the function $F : t \in [0,1] \mapsto \mu([0,t])$ is increasing and continuous on $[0,1]$.

The Poisson point process S can be written $S = \{(t_n, \lambda_n)\}_{n\ge1}$, with $|\lambda_n| \searrow 0$. Let $\{\beta_j\}_{j\ge1}$ be a sequence as found in Lemma 1.

For every $(t_n, \lambda_n) \in S$ such that $t_n \in G_j$, one sets $u_n = F^{-1}(t_n)$, and one defines the sequence l_n as $2\Big|F^{-1}\Big(B\Big(t_n, |\lambda_n|^{\beta_j}\Big)\Big)\Big|$.

Condition $\mathbf{C2}(h_{\mu,\beta})$ is said to hold when (11) holds and when $\{(u_n, l_n)\}_{n\ge1}$ forms an heterogeneous ubiquitous system with respect to $(\mu, \tau'_\mu(1))$.

We shall see in Section 7 that this holds under suitable assumptions when μ is an independent multiplicative cascade. Consequently, the assertions of Theorem 2 concerning the linear parts of the spectra apply to the process Z_W defined in (3).

Condition $\mathbf{C3}(h)$

There exists a positive Borel measure m_h on $[0,1]$ such that $m_h(\widetilde{E}^\mu_h) > 0$ and for every Borel set $E \subset [0,1]$ such that $\dim E < \tau^*_\mu(h)$, $m_h(E) = 0$.

Suppose that μ is a independent multiplicative cascade. It is shown in [10] that if the function φ_W is everywhere finite, then with probability one, condition **C3(h)** holds for all h such that $\tau_\mu^*(h) > 0$. Consequently, the assertions of Theorem 2 concerning the strictly concave parts of the spectra apply to the process Z_W defined in (3).

4 Computation of the Hausdorff spectrum of $\widetilde{X} \circ F$: Theorem 2

In this section, in order to simplify the notations, we assume that $X = \widetilde{X}$, i.e. $B(a', Q) = 0$ in (9), so that $\widetilde{X}$ and $\widetilde{Z}$ in Theorem 2 are simply denoted X and Z.

By Lemma 1, there exists a non-decreasing sequence of positive real numbers $\widetilde{\beta} = \{\beta_j\}_{j\geq 1}$ converging to β such that, with probability 1, the set $A_{\widetilde{\beta}}$ (defined in (12)) equals $\mathbb{R}_+$. Such a sequence is fixed.

4.1 Characterization of the Hölder exponents of $Z = X \circ F$

For every $j \geq 1$, for every $t \in G_j$, let $l_t = 2|F^{-1}([t - |\lambda_t|^{\beta_j}, t + |\lambda_t|^{\beta_j}])|$ and $I_t = [F^{-1}(t) - l_t, F^{-1}(t) + l_t]$. These intervals were considered in condition **C2($h_{\mu,\beta}$)** in Section 3.5. By construction of the $\{\beta_j\}_j$, one has

$$[0,1] \subset \bigcap_{J\geq 1} \bigcup_{j\geq J} \bigcup_{t\in G_j} I_t.$$

Definition 6 *Let $\alpha \geq 0$, $\delta \geq 1$ and $\varepsilon > 0$.*

A real number u_0 is said to satisfy the property $\mathcal{P}(\alpha, \delta, \varepsilon)$ if there exist an infinite number of jump points u of Z satisfying

$$|u - u_0| \leq |I_{F(u)}|^{\delta-\varepsilon} \text{ and } |I_{F(u)}|^{\alpha+\varepsilon} \leq \mu\Big(I_{F(u)}\Big) \leq |I_{F(u)}|^{\alpha-\varepsilon}. \qquad (21)$$

Remark that, by construction, if $t = F(u)$ and $t \in G_j$ for some integer $j \geq 1$, under (21) one also has $2^{-j} \leq |I_{F(u)}|^{\frac{\alpha-\varepsilon}{\beta+\varepsilon}}$ if j is large enough.

A real number u_0 is said to satisfy the property $\widetilde{\mathcal{P}}(\alpha, \delta, \varepsilon)$ if there exist an infinite number of jump points u of Z which satisfy (21) together with $|I_{F(u)}|^{\frac{\alpha+\varepsilon}{\beta-\varepsilon}} \leq 2^{-j}$ if $F(u) \in G_j$ (notice that here 2^{-j} is the size of the jump of Z at u).

One then sets for $h > 0$

$$\mathcal{T}_{\beta,h} = \left\{ u \in [0,1] : \begin{cases} \forall \varepsilon > 0,\ \exists\, \alpha \geq 0,\ \exists\, \delta \geq 1 \textit{ such that} \\ \frac{\alpha}{\beta\delta} \leq h + \varepsilon \textit{ and } u \textit{ satisfies } \mathcal{P}(\alpha,\delta,\varepsilon) \end{cases} \right\}, \tag{22}$$

$$\widetilde{\mathcal{T}}_{\beta,h} = \left\{ u \in [0,1] : \begin{cases} \forall \varepsilon > 0,\ \exists\, \alpha \geq 0,\ \exists\, \delta \geq 1 \textit{ such that} \\ \frac{\alpha}{\beta\delta} \leq h + \varepsilon \textit{ and } u \textit{ satisfies } \widetilde{\mathcal{P}}(\alpha,\delta,\varepsilon) \end{cases} \right\}. \tag{23}$$

Heuristically, the point u_0 satisfies $\mathcal{P}(\alpha,\delta,\varepsilon)$ or $\widetilde{\mathcal{P}}(\alpha,\delta,\varepsilon)$ when it is well-approximated at rate δ by intervals $I_F(u_n)$ which are selected according to the value of their μ-measure, i.e. such that $\mu(I_F(u_n))) \sim |I_F(u_n))|^\alpha$.

Remark that if $0 < h' \leq h$, then one clearly has $\mathcal{T}_{\beta,h'} \subset \mathcal{T}_{\beta,h}$.

One denotes $\overline{S} = \Big\{ t \in \mathbb{R}_+ : \exists\ \lambda \in \mathbb{R}^d,\ (t,\lambda) \in S \Big\}$, that is to say $\overline{S}$ is the projection on $\mathbb{R}_+$ of the Poisson point process S associated with $X(t)$.

This section is devoted to the proof of the following result, which is a simple consequence of next Propositions 4, 5 and 6.

Theorem 5 *Assume that* **C1** *holds. With probability one, for every $h > 0$ one has $\mathcal{A}_h \subset E^Z_h \subset \mathcal{B}_h$, where*

$$1.\ \mathcal{A}_h = \begin{cases} \widetilde{\mathcal{T}}_{\beta,h} \setminus \left[\left(\bigcup_{h' \leq h} E^\mu_{\beta h'} \right) \cup \left(\bigcup_{h' < h} \mathcal{T}_{\beta,h'} \right) \right] & \textit{if } 0 \leq h < h_{\mu,\beta} \\ \widetilde{E}^\mu_{\beta h} \setminus \left(F^{-1}(\overline{S}) \cup \bigcup_{\delta > \beta} F^{-1}(A_\delta) \right) & \textit{if } h \geq h_{\mu,\beta}. \end{cases} \tag{24}$$

$$2.\ \mathcal{B}_h = \begin{cases} \left(\mathcal{T}_{\beta,h} \setminus \bigcup_{h' < h} \widetilde{\mathcal{T}}_{\beta,h'} \right) \cup \bigcup_{h' \leq h} E^\mu_{\beta h'} & \textit{if } 0 \leq h \leq \tau'_\mu(0^+)/\beta, \\ \bigcup_{h' \geq h} \overline{E}^\mu_{\beta h'} & \textit{if } h \geq \tau'_\mu(0^+)/\beta. \end{cases} \tag{25}$$

Consequently, in order to compute the singularity spectrum of Z, it remains us to find an upper bound for $\dim \mathcal{B}_h$ and a lower bound for $\dim \mathcal{A}_h$. This is achieved in the next sections.

Proposition 4 *Assume that* **C1** *holds. With probability 1, one has:*

For every $u_0 \in [0,1]$ not a jump point of Z, let $h_\mu(u_0) = \alpha \geq 0$ and $\overline{h}_\mu(u_0) = \overline{\alpha}$, and let us write $t_0 = F(u_0) \in [0, \|\mu\|]$ and $h_X(t_0) = 1/\delta_{u_0}$ for some $\delta \geq \beta$. Then

$$\alpha/\delta_{u_0} \leq h_Z(u_0) \leq \overline{\alpha}/\delta_{u_0}. \tag{26}$$

PROOF. Let $\varepsilon > 0$. By definition of $h_\mu(u_0)$, there exists $\eta_1 > 0$ such that

$$\text{for every } 0 < r \leq \eta_1,\ \mu(B(u_0,r)) \leq r^{\alpha-\varepsilon}. \tag{27}$$

Let j_r be the unique integer such that $2^{-j_r} \leq r \leq 2^{-j_r+1}$.

By definition of $\overline{\alpha}$, one can also choose η_1 small enough so that

$$\text{for every } 0 < r \leq \eta_1, \text{ if } I \in \{I^-_{j_r+2}(u_0), I_{j_r+2}(u_0), I^+_{j_r+2}(u_0)\}, \ |\mu(I)| \geq r^{\overline{\alpha}+\varepsilon}. \tag{28}$$

Remark that $I^-_{j_r+2}(u_0) \cup I_{j_r+2}(u_0) \cup I^+_{j_r+2}(u_0) \subset B(u_0, r)$. Similarly, by definition of $h_X(t_0) = 1/\delta_{u_0}$ and Proposition 3, there exists η_2 such that

$$\text{for every number } s \text{ such that } |s| \leq \eta_2, |X(t_0+s) - X(t_0)| \leq s^{1/\delta_{u_0}-\varepsilon}, \tag{29}$$

and for some sequence $(h_j)_{j\geq 1}$ such that $|h_j| \searrow 0$,

$$|X(t_0+h_j) - X(t_0)| \geq |h_j|^{1/(\delta_{u_0}+\varepsilon)}. \tag{30}$$

Since the function F is continuous on $[0,1]$, one can thus choose η_1 small enough so that $F(B(u_0,\eta_1)) \subset B(t_0,\eta_2)$.

• Let $-\eta_1 \leq r \leq \eta_1$. By (29) and then (27), one has

$$\begin{aligned} |Z(u_0+r) - Z(u_0)| &= |X \circ F(u_0+r) - X \circ F(u_0)| \\ &\leq |F(u_0+r) - F(u_0)|^{1/\delta_{u_0}-\varepsilon} \leq |r|^{(\alpha+\varepsilon)/\delta_{u_0} - (\alpha+\varepsilon)\varepsilon}. \end{aligned}$$

since $|F(u_0+r) - F(u_0)| \leq \mu(B(u_0,|r|))$. This holds for every $\varepsilon > 0$, hence the lower bound of (26).

• Let j be such that (30) holds, and let r_j be the unique real number such that $F(u_0+r_j) = t_0 + h_j$. One has

$$|Z(u_0+r_j) - Z(u_0)| = |X(t_0+h_j) - X(t_0)| \geq |h_j|^{1/(\delta_{u_0}+\varepsilon)}.$$

By (28), $\mu([u_0, u_0+r_j]) \geq \mu(I^+_{j_{r_j}+2}(u_0)) \geq |r_j|^{\overline{\alpha}+\varepsilon}$. Since $F(u_0+r_j) - F(u_0) = h_j$, one gets $|h_j| \geq |r_j|^{\overline{\alpha}+\varepsilon}$, and thus

$$|Z(u_0+r_j) - Z(u_0)| \geq |r_j|^{(\overline{\alpha}+\varepsilon)/\delta_{u_0} + (\overline{\alpha}+\varepsilon)\varepsilon}.$$

Since this holds for an infinite number of r_j converging to zero and then for every $\varepsilon > 0$, one gets the conclusion.

Proposition 5 *Assume that* $u_0 \in \widetilde{\mathcal{T}}_{\beta,h}$ *for some* $h \geq 0$*. Then* $h_Z(u_0) \leq h$.

PROOF. Let $\varepsilon > 0$. The proof uses the following Lemma of [21].

Lemma 5 *Let us assume that a function f is discontinuous on a dense set of $\mathbb{R}$. For a fixed $x \in \mathbb{R}$, let us assume that there exists a sequence $\{r_n\}_n$ converging to x such that for every n, f has right and left limits $f(r_n^+)$ and $f(r_n^-)$ at r_n, and $|f(r_n^+) - f(r_n^-)| = s_n > 0$. Then*

$$h_f(x) \leq \liminf_{n\to+\infty} \frac{|\log s_n|}{|\log |r_n - x_0||}.$$

Let $(u_n)_{n\geq 1}$ be an infinite sequence of jump points of Z that verifies (21) for u_0 as well as the fact that the size of the jump of Z at u_n is greater than $|I_{F(u_n)}|^{\frac{\alpha+\varepsilon}{\beta-\varepsilon}}$. Lemma 5 yields then

$$h_Z(u_0) \leq \liminf_{n\to+\infty} \frac{|\log |I_{F(u_n)}|^{\frac{\alpha+\varepsilon}{\beta-\varepsilon}}|}{|\log |I_{F(u_n)}|^{\delta}|} \leq \frac{\alpha+\varepsilon}{\delta(\beta-\varepsilon)}.$$

This remains true for an infinite number of jump points u_n converging to u_0, and letting ε go to zero gives the result.

Proposition 6 *Assume that* **C1** *holds. With probability 1, one has the following property: For every $u_0 \in [0,1]$ not a jump point of Z, if $h_Z(u_0) < h_\mu(u_0)/\beta$, then $u_0 \in \mathcal{T}_{\beta,h_Z(u_0)}$.*

PROOF. One sets $h = h_Z(u_0)$, $\alpha = h_\mu(u_0)$, $t_0 = F(u_0)$ and $h_X(t_0) = 1/\delta_{u_0}$ for some $\delta_{u_0} \geq \beta$. One necessarily has $\delta_{u_0} > \beta$ otherwise, if $\delta_{u_0} = \beta$, by Proposition 4 one would have $h \geq \alpha/\beta$.

Let $\varepsilon > 0$. By definition of h, there exists a sequence $(r_n)_{n\geq 1}$ such that $|r_n| \searrow 0$ and $|Z(u_0+r_n) - Z(u_0)| \geq |r_n|^{h+\varepsilon}$. We set $u_n = u_0 + r_n$, and $t_n = F(u_n)$. One has $|X(t_n) - X(t_0)| \geq |r_n|^{h+\varepsilon}$, and $|t_n - t_0| = \mu([u_0, u_n]) \leq \mu(B(u_0, |r_n|)) \leq |r_n|^{\alpha-\varepsilon}$ by (27).

Let us denote by j_n the unique integer such that $2^{-j_n} \leq |t_n - t_0| < 2^{-j_n+1}$. Let $\varepsilon' > 0$ and let us write

$$X(t_n) - X(t_0) = \sum_{j<[j_n/(\beta+\varepsilon')]} X_j(t_n) - X_j(t_0) + \sum_{j\geq[j_n/(\beta+\varepsilon')]} X_j(t_n) - X_j(t_0).$$

By Proposition 3, there exists $\varepsilon' > 0$ small enough such that (13) holds. One thus has

$$\begin{aligned}\sum_{j<[j_n/(\beta+\varepsilon')]} |X_j(t_n)-X_j(t_0)| &\geq \Big| \sum_{j<[j_n/(\beta+\varepsilon')]} X_j(t_n)-X_j(t_0) \Big| \\ &\geq |X(t_n)-X(t_0)| - |t_n-t_0|^{1/(\beta+\varepsilon)} \\ &\geq |r_n|^{h+\varepsilon} - |r_n|^{\frac{\alpha-\varepsilon}{\beta+\varepsilon}}.\end{aligned}$$

The parameter ε can be chosen small enough so that $(h+\varepsilon)(\beta+\varepsilon)<\alpha-\varepsilon$, hence

$$\sum_{j<[j_n/(\beta+\varepsilon')]} |X_j(t_n)-X_j(t_0)| \geq C|r_n|^{h+\varepsilon}. \tag{31}$$

Remembering (15), we conclude that $\sum_{j<[j_n/(\beta+\varepsilon')]} X_j(\cdot)$ has a jump point between t_n and t_0 (since the contribution of the drift is not large enough to explain (31)).

Let us consider one among the jump points with tallest size, i.e. a real number T_n of $[t_0,t_n]$ such that T_n is a jump point for X_{J_n} for some J_n and there is no jump point of $X(t)$ in $[t_0,t_n]$ belonging to some $G_{j'}$, $j'<J_n$. Remark that since $h_X(t_0)=1/\delta_{u_0}$, one has for n large enough $j_n/(\delta_{u_0}+\varepsilon)\leq J_n \leq j_n/(\beta+\varepsilon')$.

We now apply Lemma 2 with $\delta=\delta_{u_0}$. One chooses j_n large enough so that ε_{j_n} and η_{j_n} are less than $\varepsilon/2$. Let k be the unique integer such that $t_0\in[k2^{-j_n},(k+1)2^{-j_n})$. One has $[t_0,t_n]\subset I=\bigcup_{l=k-2,\dots,k+2} I_{j_n,l}$. By Lemma 2 applied to the five intervals contained in I, the number of jumps in the interval $[t_0,t_n]$ of all the X_j's, $j<\left[\frac{j_n}{\beta+\varepsilon'}\right]$, is less than $5.2^{j_n\eta_{j_n}}$.

Using (31) and the existence of T_n, one gets that

$$|D| + 52^{j_n\eta_{j_n}}2^{-J_n} \geq \sum_{j<[j_n/\beta+\varepsilon']} |X_j(t_n)-X_j(t_0)| \geq C|r_n|^{h+\varepsilon},$$

where D stands for the contribution of the drift of all the X_j's, $j<\left[\frac{j_n}{\beta+\varepsilon'}\right]$, on the interval $[t_0,t_n]$. But, again by (15), $|D|\leq |t_n-t_0|^{1/(\beta+\varepsilon)}\leq |r_n|^{\frac{\alpha-\varepsilon}{\beta+\varepsilon}}$. As above, since $\frac{\alpha-\varepsilon}{\beta+\varepsilon}>h$, one has for n large enough $52^{j_n\eta_{j_n}}2^{-J_n}\geq C|r_n|^{h+\varepsilon}$, for another constant C.

Finally, since **C1** yields $j_n=O\Big(|\log(|r_n|)|\Big)$ and η_{j_n} goes to zero when $n\to+\infty$, one obtains

$$2^{-J_n} \geq C|r_n|^{h+2\varepsilon} \geq |r_n|^{h+3\varepsilon}. \tag{32}$$

Let us denote by U_n the real number $F^{-1}(T_n)$, and consider $I_{T_n}=I_{F(U_n)}$ (the intervals I_t for $t\in G_j$ were defined at the beginning of Section 4.1). By construction this interval satisfies $\mu(I_{T_n})\geq 2\cdot 2^{-J_n\beta_{J_n}}$. One has $u_0\in I_{T_n}$ for n large enough because $\beta_{J_n}J_n\leq \frac{\beta_{J_n}}{\beta+\varepsilon'}j_n<j_n$. Thus by (27) $2\cdot 2^{-J_n\beta_{J_n}}\leq\mu(I_{T_n})\leq |I_{T_n}|^{\alpha-\varepsilon}$ for n large enough. Let us write $\mu(I_{T_n})=|I_{T_n}|^{\alpha_n}$ for some $\alpha_n\geq\alpha-2\varepsilon$.

On the other side, one knows that $|u_0 - U_n| \leq |r_n|$. But $|r_n| \leq 2^{-J_n \frac{1}{h+3\varepsilon}} \leq C|I_{T_n}|^{\frac{\alpha_n}{\beta_{J_n}(h+3\varepsilon)}}$ by (32). Let us define $\delta_n = \frac{\alpha_n}{\beta_{J_n}(h+3\varepsilon)}$. For ε small enough and n large enough, one sees that $\delta_n \geq 1$ (since $h < \alpha/\beta$).

If γ_1 is the constant of condition **C1**, for every n large enough, the couple (α_n, δ_n) belongs to the square $[0, \gamma_1] \times [1, \delta_{u_0} + \varepsilon]$. Without loss of generality by extracting a subsequence, we can assume that (α_n, δ_n) converges to (α_0, δ_0). By construction $\frac{\alpha_0}{\beta\delta_0} \leq h + 4\varepsilon$. Hence $\mathcal{P}(\alpha_0, \delta_0, 4\varepsilon)$ holds.

PROOF. of Theorem 5. Let $h \geq 0$ and $u_0 \in E_h^Z$. By Propositions 5 and 6, $u_0 \in \bigcup_{h' \leq h} E^{\mu}_{\beta h'} \cup \mathcal{T}_{\beta,h} \setminus \bigcup_{h'<h} \widetilde{\mathcal{T}}_{\beta,h'}$. Also, by Proposition 4 $u_0 \in \bigcup_{h' \geq h} \overline{E}^{\mu}_{\beta h'}$. Consequently $E_h^Z \subset \mathcal{B}_h$.

Propositions 5 and 6 clearly imply that $\widetilde{\mathcal{T}}_{\beta,h} \setminus \left[\left(\bigcup_{h' \leq h} E^{\mu}_{\beta h'} \right) \cup \left(\bigcup_{h'<h} \mathcal{T}_{\beta,h'} \right) \right] \subset E_h^Z$. Thus $\mathcal{A}_h \subset E_h^Z$ when $h < h_{\mu,\beta}$.

Finally, when $h \geq h_{\mu,\beta}$, if $u_0 \in \mathcal{A}_h$, by Proposition 4 $h_Z(u_0) = \beta h_\mu(u_0)/\beta = h_\mu(u_0)$ (since $h_\mu(u_0) = \overline{h}_\mu(u_0)$). Hence $\mathcal{A}_h \subset E_h^Z$.

4.2 Upper bound for the singularity spectrum of Z

Proposition 7 *With probability 1, for every $h \geq \tau_\mu'(0^+)/\beta$, $\dim E_h^Z \leq \tau_\mu^*(\beta h)$ and $E_h^Z = \emptyset$ if $h > \alpha_{\max}/\beta$.*

PROOF. This follows from Theorem 5, item 2. and Proposition 1, items (3) and (4).

In order to get an upper bound for the increasing part of the multifractal spectrum of Z, some notations and new sets are needed.

For every $j \geq 1$, $t \in G_j$ and $\delta \geq 1$, let

$$I_t^{(\delta)} = B(F^{-1}(t), l_t^{\delta}). \tag{33}$$

Let us consider, for $\alpha \geq 0$, $\varepsilon > 0$ and $\delta \geq 1$, the sets

$$T_{\alpha,\delta,\varepsilon} = \bigcap_{J \geq 1} \bigcup_{j \geq J} \bigcup_{t \in G_j :\, |I_t|^{\alpha+\varepsilon} \leq \mu(I_t) \leq |I_t|^{\alpha-\varepsilon}} B_t^{(\delta)}. \tag{34}$$

The Hausdorff dimension of the sets $T_{\alpha,\delta,\varepsilon}$ is easily tractable (as shown by the following proposition). Moreover, these sets are closely related with the sets $\mathcal{T}_{\beta,h}$.

Lemma 6 *Assume that* **C1** *holds for* μ. *For every* $\alpha > 0$ *such that* $\tau_\mu^*(\alpha) \geq 0$, $\delta \geq 1$ *and* $\varepsilon > 0$

$$\dim T_{\alpha,\delta,\varepsilon} \leq \frac{\sup_{\alpha' \leq \alpha+\varepsilon} \tau_\mu^*(\alpha) + \varepsilon}{\delta}. \tag{35}$$

PROOF.

We first use Lemma 4. Due to the definition of I_t, the weak redundancy property of $S = \bigcup_{j\geq 0}\{(t, |\lambda_t|^{\beta_j}) : t \in G_j\}$ implies the existence of a non-negative sequence $(\xi_j)_{j\geq 0}$ converging to 0 such that as soon as $G_j \neq \emptyset$, the set $\{I_t : t \in G_j\}$ can be written as a union of $2^{j\xi_j}$ families $\mathcal{G}_{j,i}$ of pairwise disjoint intervals.

One has $T_{\alpha,\delta,\varepsilon} = \bigcap_{J\geq 1} \bigcup_{j\geq J} S_j$, where

$$S_j = \bigcup_{t\in G_j :\, |I_t|^{\alpha+\varepsilon} \leq \mu(I_t) \leq |I_t|^{\alpha-\varepsilon}} I_t^{(\delta)}. \tag{36}$$

Fix $\alpha_0 \in (0, \tau_\mu'(0^+))$. Let $\alpha \in [\alpha_0, \tau_\mu'(0^+))$ and $\varepsilon \in (0, \alpha_0/2)$. Let $J \geq 0$ and $j \geq J$. Let $t \in G_j$ and let J_t denotes the unique integer such that $2^{-J_t} \leq |I_t| \leq 2^{-J_t+1}$. If $|I_t|^{\alpha+\varepsilon} \leq \mu(I_t) \leq |I_t|^{\alpha-\varepsilon}$, then at least one of the intervals $I_{J_t+2,k}$ such that $I_{J_t+2,k} \cap I_t \neq \emptyset$ must satisfy $\mu(I_{J_t+2,k}) \geq \frac{1}{16}|I_t|^{\alpha+\varepsilon} \geq C2^{-(J_t+2)(\alpha+\varepsilon)}$, where C is a constant depending only on α. Moreover, due to **C1** and the definition of the interval I_t, there exists two positive constants γ and γ' independent of t such that for j large enough one has $\gamma j \leq J_t + 2 \leq \gamma' j$.

For every integer $m \geq 1$, let $F_m = \{I_{m,k} : \mu(I_{m,k}) \geq C2^{-m(\alpha+\varepsilon)}\}$ for every i. We deduce from the last considerations that every I_t belonging to some $\mathcal{G}_{j,i}$ and satisfying $\mu(I_t) \geq |I_t|^{\alpha+\varepsilon}$ must intersect an element I of $\bigcup_{\gamma j \leq m \leq \gamma' j} F_m$. In this case, $|I|^\delta \leq |I_t^{(\delta)}| \leq C|I|^\delta$ for some constant C depending only on δ.
Moreover, since the elements of $\mathcal{G}_{j,i}$ are pairwise disjoint, the intervals I of $\bigcup_{\gamma j \leq m \leq \gamma' j} F_m$ previously selected intersect at most two elements of $\mathcal{G}_{j,i}$. Also, we learn from Proposition 2 that for m large enough, the cardinality of F_m is less than or equal to $2^{m \sup_{\alpha' \leq \alpha+\varepsilon} \tau_\mu^*(\alpha')+\varepsilon}$.

Now let $s > \left(\sup_{\alpha' \leq \alpha+\varepsilon} \tau^*(\alpha) + \varepsilon\right)/\delta$. Recall Definition 3. It follows from the previous remarks that for some constant $C' > 0$,

$$
\begin{aligned}
\mathcal{H}^s_{C'2^{-\gamma J}}(T_{\alpha,\delta,\varepsilon}) &\leq \sum_{j\geq J} \sum_{t\in G_j:\,|I_t|^{\alpha+\varepsilon}\leq \mu(I_t)\leq |I_t|^{\alpha-\varepsilon}} |I_t^{(\delta)}|^s \\
&= \sum_{j\geq J}\sum_i \sum_{I_t\in \mathcal{G}_{j,i}:|I_t|^{\alpha+\varepsilon}\leq \mu(I_t)} |I_t^{(\delta)}|^s \leq \sum_{j\geq J}\sum_i \sum_{\gamma j\leq m\leq \gamma' j} 2\sum_{I\in F_m} C|I|^{s\delta} \\
&\leq 2C\sum_{j\geq J} 2^{j\xi_j} \sum_{\gamma j\leq m\leq \gamma' j} 2^{-s\delta m} 2^{m\sup_{\alpha'\leq \alpha+\varepsilon}\tau_\mu^*(\alpha')+\varepsilon},
\end{aligned}
$$

Since $\xi_j \to 0$ when $j\to+\infty$, $\lim_{J\to\infty}\mathcal{H}^s_{C'2^{-\gamma J}}(T_{\alpha,\delta,\varepsilon})=0$, thus $\dim T_{\alpha,\delta,\varepsilon}\leq s$.

Proposition 8 *Assume that* **C1** *holds. With probability 1, for every exponent* $h\in[0,\tau_\mu'(0^+)/\beta)$, $\dim E_h^Z \leq D_{\mu,\beta}(h)$.

PROOF. If $h=0$, then it follows from Proposition 4 that E_h^Z is contained in the set $F^{-1}(\overline{S})\cup E_0^\mu \cup (\cap_{\delta>1}A_\delta)$. Thus $\dim E_h^Z=0$.

Let us now fix $h\in(0,\frac{\tau_\mu'(0^+)}{\beta})$. Item 2. of Theorem 5 implies that $\dim E_h^Z \leq \max\Big(\dim \mathcal{T}_{\beta,h}\setminus\bigcup_{h'<h}\widetilde{\mathcal{T}}_{\beta,h'}, \dim\bigcup_{\alpha\leq\beta h}E_\alpha^\mu\Big)$. Item 2. of Proposition 1 yields $\dim\bigcup_{\alpha\leq\beta h}E_\alpha^\mu \leq \tau_\mu^*(\beta h)$. It remains us to find an upper bound for $\dim\mathcal{T}_{\beta,h}$.

For every $\varepsilon>0$, $\mathcal{T}_{\beta,h}\subset\bigcup_{\substack{(\alpha,\delta)\in\mathbb{Q}\times\mathbb{Q}\\ \alpha>0,\,\tau_\mu^*(\alpha)\geq 0,\,\delta\geq 1,\,\alpha/\beta\delta\leq h+\varepsilon}} T_{\alpha,\delta,\varepsilon}$. Lemma 6 yields

$$
\begin{aligned}
\dim \mathcal{T}_{\beta,h} &\leq \sup_{\substack{(\alpha,\delta)\in\mathbb{Q}\times\mathbb{Q}\\ \alpha>0,\,\tau_\mu^*(\alpha)\geq 0,\,\delta\geq 1,\,\alpha/\beta\delta\leq h+\varepsilon}} \dim T_{\alpha,\delta,\varepsilon} \\
&\leq \sup_{\substack{(\alpha,\delta)\in\mathbb{Q}\times\mathbb{Q}\\ \alpha\geq 0,\,\tau_\mu^*(\alpha)\geq 0,\,\delta\geq 1,\,\alpha/\beta\delta\leq h+\varepsilon}} \frac{\sup_{\alpha'\leq\alpha+\varepsilon}\tau_\mu^*(\alpha')+\varepsilon}{\delta} \\
&\leq \max(\beta(h+\varepsilon)d_1(h,\varepsilon), d_2(h,\varepsilon)),
\end{aligned}
$$

where $\begin{cases} d_1(h,\varepsilon)=\sup_{\alpha\geq\beta h}\frac{\sup_{\alpha'\leq\alpha+\varepsilon}\tau_\mu^*(\alpha')+\varepsilon}{\alpha}, \\ d_2(h,\varepsilon)=\sup_{0\leq\alpha<\beta h,\,\tau_\mu^*(\alpha)\geq 0,\,\delta\geq 1,\,\alpha/\beta\delta\leq h+\varepsilon}\frac{\sup_{\alpha'\leq\alpha+\varepsilon}\tau_\mu^*(\alpha')+\varepsilon}{\delta}. \end{cases}$

Since $\beta h\leq\tau_\mu'(0^+)$, $\lim_{\varepsilon\to 0}d_2(h,\varepsilon)=\tau_\mu^*(\beta h)$.
The next observations are already done in [8] (they are easy to check using the continuity of τ_μ^* on its support and the fact that $\sup_{\alpha\geq 0:\,\tau_\mu^*(\alpha)\geq 0}\tau_\mu^*(\alpha)/\alpha$ is reached for $\alpha=\tau_\mu'(1^-)$):

- If $h\leq\tau_\mu'(1)/\beta$, then $\lim_{\varepsilon\to 0}d_1(h,\varepsilon)=1$.

- If $h\geq\tau_\mu'(1)/\beta$, then $\lim_{\varepsilon\to 0}d_1(h,\varepsilon)=\tau_\mu^*(\beta h)/\beta h$.

We finally get the desired upper bound for $\dim\mathcal{T}_{\beta,h}$ and thus also for $\dim E_h^Z$.

Proposition 9 *Suppose that* **C1** *holds. With probability one, for every* $h \geq h_{\mu,\beta}$ *such that* $\mathbf{C3}(\beta h)$ *holds, one has* $\dim E_h^Z \geq \tau_\mu^*(\beta h)$.

PROOF. Fix a realization of Z and $h \geq h_{\mu,\beta}$ such that $\mathbf{C3}(\beta h)$ holds.

Let $m_{\beta h}$ be the measure given by $\mathbf{C3}(\beta h)$. Combining $\mathbf{C3}(\beta h)$ and item 1. of Theorem 5, it is enough to prove that $m_{\beta h}\Big(\bigcup_{\delta>\beta} E_\delta\Big) = 0$ and $m_{\beta h}\Big(\widetilde{E}^\mu_{\beta h} \cap F^{-1}(\overline{S})\Big) = 0$, where $E_\delta = \widetilde{E}^\mu_{\beta h} \cap (F^{-1}(A_\delta) \setminus F^{-1}(\overline{S}))$.

Since S is countable and the family of sets A_δ is monotonic, it remains to show that dim $E_\delta < \tau^*(\beta h)$ for every $\delta > \beta$. Fix such a δ and let $u \in E_\delta$.

Let $\delta_{F(u)} = \limsup_{j\to\infty} \sup_{t\in G_j} \frac{\log|t-F(u)|}{\log|\lambda_t|}$. Since $F(u) \in A_\delta$ one has $\delta_{F(u)} \geq \delta$. Let $(t_n)_{n\geq 1}$ be a sequence of points of S verifying $\lim_{n\to\infty} \frac{\log|t_n-F(u)|}{|\lambda_{t_n}|} = \delta_{F(u)}$.

Denote $u_n = F^{-1}(t_n)$. Since $u \in \widetilde{E}_{\beta h}$, one has

$$\limsup_{n\to\infty} \frac{\log|u-u_n|}{\log|I_{t_n}|} = \frac{1}{\beta h} \limsup_{n\to\infty} \frac{|F(u)-F(u_n)|}{\log|I_{t_n}|}.$$

Moreover, since $u \in I_{t_n} \cap \widetilde{E}_{\beta h}$, one also has $\lim_{n\to\infty} \frac{\log|I_{t_n}|}{\log|F(I_{t_n})|} = \frac{1}{\beta h}$. But by construction of the I_{t_n}'s one has $\lim_{n\to\infty} \frac{\log|F(I_{t_n})|}{\log|\lambda_{t_n}|} = \beta$. Consequently,

$$\limsup_{n\to\infty} \frac{\log|u-u_n|}{\log|I_{t_n}|} = \frac{\delta_{F(u)}}{\beta} \geq \frac{\delta}{\beta} > 1.$$

It follows from these remarks that $E_\delta \subset T_{\beta h,\delta/\beta,\varepsilon}$ for all $\varepsilon > 0$. Lemma 6 yields that dim $E_\delta \leq \beta\tau^*(\beta h)/\delta < \tau^*(\beta h)$.

Proposition 10 *Suppose that* **C1** *and* $\mathbf{C2}(h_{\mu,\beta})$ *hold. Then, with probability one, for every* $\delta > 1$, $\dim E^Z_{\tau'_\mu(1)/(\beta\delta)} \geq \tau'(1)/\delta$; *equivalently, for every* $0 < h < h_\beta$, $\dim E_h^Z = d_Z(h) \geq \beta h$.

PROOF. Let $\delta > 1$, $h = h_{\mu,\beta}/\delta$ and $d = \tau'_\mu(1)/\delta$.

Fix a realization of Z and S such that the properties involved in condition $\mathbf{C2}(h_{\mu,\beta})$ are satisfied. Theorem 4 provides us with the non-decreasing sequence $\widetilde{\delta}$ converging to δ, the positive sequence $\widetilde{\varepsilon}$ converging to 0, the set $S_\mu(\widetilde{\delta}, \tau'_\mu(1), \widetilde{\varepsilon})$, and the measure m_δ.

By construction, all the points of $S_\mu(\widetilde{\delta}, \tau'_\mu(1), \widetilde{\varepsilon})$ satisfy $\widetilde{\mathcal{P}}(\tau'_\mu(1), \delta, \varepsilon)$ for all $\varepsilon > 0$. So $S_\mu(\widetilde{\delta}, \tau'_\mu(1), \widetilde{\varepsilon}) \subset \widetilde{\mathcal{T}}_{\beta,h}$. Moreover, $m_\delta(S_\mu(\widetilde{\delta}, \tau'_\mu(1), \widetilde{\varepsilon})) > 0$, which, by Theorem 4, implies that dim $S_\mu(\widetilde{\delta}, \tau'_\mu(1), \widetilde{\varepsilon}) \geq \tau'(1)/\delta = \beta h$.

When proving Proposition 8, we established that every set of the non-decreasing sequence $(\mathcal{T}_{\beta,h'})_{h'<h}$ is of Hausdorff dimension less than βh. Thus $m_\delta(\cup_{h'<h}\mathcal{T}_{\beta,\delta}) = 0$, and thus $m_\delta(S_\mu(\widetilde{\delta}, \tau'_\mu(1), \widetilde{\varepsilon}) \setminus \cup_{h'<h}\mathcal{T}_{\beta,\delta}) > 0$. Using Theorem 5(1) and the fact that $S_\mu(\widetilde{\delta}, \tau'_\mu(1), \widetilde{\varepsilon}) \subset \widetilde{\mathcal{T}}_{\beta,h}$, we get that $m_\delta(E^Z_h) > 0$, hence the conclusion.

5 The case $a' \neq 0$ and $Q = 0$: Item *2.* of Theorem 3

In this section, we use the decomposition (9) with $a' \neq 0$ and $Q = 0$ to write $Z(t) = \widetilde{X}(F(t)) + F(t)a$, with $a \in \mathbb{R} \setminus \{0\}$. Let us write $\widetilde{Z} = \widetilde{X} \circ F$.

Since $h_F(u) = h_\mu(u)$ for every point $u \in [0, 1]$, equation (4) implies that $h_Z(u) \geq \min\left(h_{\widetilde{Z}}(u), h_F(u)\right) = \min\left(h_{\widetilde{Z}}(u), h_\mu(u)\right)$. Consequently, for all $h \geq 0$ one has

$$E^Z_h \subset \bigcup_{h' \leq h} E^{\widetilde{Z}}_{h'} \cup E^\mu_{h'}.$$

By using Theorem 5(2), Proposition 1 and the estimates obtained in the proof of Proposition 8, one concludes that

$$\forall\ h \geq 0,\ d_Z(h) \leq \max\left(D_{\mu,\beta}(h), \tau^*_\mu(h)\right) = \begin{cases} \widetilde{D}_{\mu,\beta}(h) & \text{if } \beta < 1, \\ D_{\mu,\beta}(h) & \text{otherwise.} \end{cases}$$

This yields the upper bound of the singularity spectrum claimed in Theorem 2. The following remarks yield the lower bound.

- Suppose that $\beta < 1$. If $h \geq \widetilde{h}_{\mu,\beta}$, then $\tau^*_\mu(h) \geq D_{\mu,\beta}(h)$. Let $h > \widetilde{h}_{\mu,\beta}$. By Proposition 4, for every $u \in \widetilde{E}^\mu_h$, one has $h_\mu(u) < h_{\widetilde{Z}}(h)$, which yields $h_Z(u) = h_\mu(u)$. Therefore, $\widetilde{E}^\mu_h \subset E^Z_h$, and $\dim E^Z_h \geq \tau^*_\mu(h)$ under **C3(h)**. If now $h = \widetilde{h}_{\mu,\beta}$ and $D_{\mu,\beta}(\widetilde{h}_{\mu,\beta}) < \tau^*_\mu(\widetilde{h}_{\mu,\beta})$, the procedure is the same as for $h > \widetilde{h}_{\mu,\beta}$.

We now use **C2($h_{\mu,\beta}$)** and the same limsup-set $S_\mu(\widetilde{\delta}, \tau'_\mu(1), \widetilde{\varepsilon})$ as in the proof of Proposition 10.

If $h = \widetilde{h}_{\mu,\beta}$ and $D_{\mu,\beta}(\widetilde{h}_{\mu,\beta}) = \tau^*_\mu(\widetilde{h}_{\mu,\beta})$, let $\delta = h_{\mu,\beta}/\widetilde{h}_{\mu,\beta}$. Lemma 5 combined with the continuity of F yield that the limsup-set $S_\mu(\widetilde{\delta}, \tau'_\mu(1), \widetilde{\varepsilon}) \setminus (\cup_{h'<h}\mathcal{T}_{\beta,h'})$ provided by **C2($h_{\mu,\beta}$)** is included in $E^Z_{\widetilde{h}_{\mu,\beta}}$. Since $\delta > 1$, the conclusion follows.

If $0 < h < \widetilde{h}_{\mu,\beta}$, then $\tau_\mu^*(h) < D_{\mu,\beta}(h)$ (because of the concavity of τ_μ^*). Then the same argument as when $h = \widetilde{h}_{\mu,\beta}$ and $D_{\mu,\beta}(\widetilde{h}_{\mu,\beta}) = \tau_\mu^*(\widetilde{h}_{\mu,\beta})$ holds with $\delta = h_{\mu,\beta}/h$.

• Suppose that $\beta \geq 1$. The case $h < h_{\mu,\beta}$ is treated as the case $h < \widetilde{h}_{\mu,\beta}$ when $\beta < 1$. If $h \geq h_{\mu,\beta}$, using again Lemma 5 and the continuity of F, one has $\widetilde{E}^\mu_{\beta h} \setminus \left(F^{-1}(\overline{S}) \cup \bigcup_{\delta>\beta} F^{-1}(A_\delta)\right) \subset E^Z_h$. One concludes as in the proof of Proposition 9.

6 The case $Q \neq 0$: Item *3.* of Theorem 3

Let us begin with a proposition which takes care of the Brownian part $B \circ F$.

Proposition 11 *Let μ be a positive measure on $[0,1]$ and $B_{1/2}$ a Brownian motion. With probability 1, $\forall u_0 \in [0,1]$, $h_\mu(u_0)/2 \leq h_{B_{1/2}\circ F}(u_0) \leq \overline{h}_\mu(u_0)/2$.*

PROOF. Let $\varepsilon > 0$. For almost every sample path of $B_{1/2}$, one has

$$\forall t_0,\ \forall t \text{ close enough to } t_0,\ |B_{1/2}(t) - B_{1/2}(t_0)| \leq |t - t_0|^{1/2-\varepsilon}, \tag{37}$$

and there is an infinite number of t_n converging to t_0 such that

$$|B_{1/2}(t) - B_{1/2}(t_0)| \geq |t - t_0|^{1/2+\varepsilon}. \tag{38}$$

Let $u_0 \in [0,1]$. For u close enough to u_0, (37) implies that

$$|B_{1/2} \circ F(u) - B_{1/2} \circ F(u_0)| \leq |F(u) - F(u_0)|^{1/2-\varepsilon} \leq |u - u_0|^{(h_\mu(u_0)-\varepsilon)(1/2-\varepsilon)}.$$

for some constant C. Moreover, by (38) there is an infinite number of points $u_n = F^{-1}(t_n)$ such that

$$|B_{1/2} \circ F(u_n) - B_{1/2} \circ F(u_0)| \geq |F(u_n) - F(u_0)|^{1/2+\varepsilon} \geq |u_n - u_0|^{(\overline{h}_\mu(u_0)+\varepsilon)(1/2+\varepsilon)}.$$

The result follows.

As a consequence of Proposition 11 one has the following result (see [37] and references therein for results of the same kind on $B \circ \mu$).

Proposition 12 *Let μ be a positive Borel measure on $[0,1]$, let $B_{1/2}$ be a Brownian motion. With probability 1, for every $h \geq 0$, $d_{B\circ F}(h) \leq \tau_\mu^*(2h)$ and $E_h^{B\circ F} = \emptyset$ if $\tau_\mu^*(2h)$. Moreover, if* **C3(**$2h$**)** *holds, $d_{B\circ F}(h) = \tau_\mu^*(2h)$.*

PROOF. Let $h \geq \tau_\mu^*(0^+)/2$. By Proposition 11, $E_h^{B\circ F} \subset \bigcup_{h' \geq 2h} \overline{E}_{h'}^\mu$, and by Proposition 1 $\dim \bigcup_{h' \geq 2h} \overline{E}_{h'}^\mu \leq \tau_\mu^*(2h)$.

Let $h \leq \tau_\mu^*(0^+)/2$. By Proposition 11, $E_h^{B\circ F} \subset \bigcup_{h' \leq 2h} E_{h'}^\mu$, and by Proposition 1 one gets $\dim \bigcup_{h' \leq 2h} E_{h'}^\mu \leq \tau_\mu^*(2h)$.

If **C3(**$2h$**)** holds, $\widetilde{E}_{2h}^\mu \subset E_h^{B\circ F}$ and $\dim \widetilde{E}_{2h}^\mu = \tau_\mu^*(2h)$.

Theorem 2., item **3.** is obtained using the same arguments as in Section 5.

7 Back to the fixed points of the smoothing transformation (1)

7.1 Recalls on Mandelbrot multiplicative cascades μ, and some self-similarity properties of $X \circ \mu$

Let us recall how the measure μ_W on $[0,1]$ is obtained. Let $\mathcal{A}$ be the alphabet $\{0, \dots, b-1\}$ and $\mathcal{A}^* = \bigcup_{n \geq 0} \mathcal{A}^n$ ($\mathcal{A}^0$ contains the empty word $\emptyset$). Consider a sequence $\Big((W_0(w), \dots, W_{b-1}(w))\Big)_{w \in \mathcal{A}^*}$ of independent copies of W. For $n \geq 1$, let $\mu_{W,n}$ be the measure defined on $[0,1]$ by uniformly distributing on every b-adic interval of the form $\Big[\sum_{k=1}^n w_k b^{-k}, b^{-n} + \sum_{k=1}^n w_k b^{-k}\Big]$, $w_1 w_2 \cdots w_n \in \mathcal{A}^n$, the mass $W_{w_1}(\emptyset) \cdot W_{w_2}(w_1) \cdots W_{w_n}(w_1 w_2 \cdots w_{n-1})$. Then, with probability one, the sequence of multiplicative cascades $(\mu_{W,n})_{n \geq 1}$ converges weakly on $[0,1]$, as $n \to \infty$, to a measure μ_W called the independent multiplicative cascade measure associated with W.

The real number $\varphi_W'(1)$ has a geometric interpretation: Both the lower and upper Hausdorff dimensions (for their definition, see [36,26]).

Let us consider such a measure $\mu = \mu_W$, and assume that μ and the Lévy process X are independent. The probability space $(\Omega, \mathbb{P})$ can be written as a product $(\Omega_S \times \Omega_\mu, \mathbb{P}_S \otimes \mathbb{P}_\mu)$, where $(\Omega_S, \mathbb{P}_S)$ and $(\Omega_\mu, \mathbb{P}_\mu)$ are the probability spaces on which are respectively defined the Poisson point process S and the measure μ.

If, moreover, $X = X_\beta$ and $\mu = \mu_{W_\beta}$ as in Section 1, the reader can check that the following property holds: $\forall n \geq 1$

$$\Big(Z_{W,(k+1)b^{-n}} - Z_{W,kb^{-n}}\Big)_{0\leq k<b^{-n}} \stackrel{d}{\equiv} \Big(Z(w)\prod_{k=1}^{n} W_{w_k}(w_1\cdots w_{k-1})\Big)_{w\in\mathcal{A}^n}, \quad (39)$$

where, on the right hand side,
- the set $\mathcal{A}^n$ is described in lexicographical order,
- the random vectors $(W_0(w),\dots,W_{b-1}(w))$'s are i.i.d. with W,
- the random values $Z(w)$'s are i.i.d. with $Z_{W,1}$ and are independent of the $(W_0(w),\dots,W_{b-1}(w))$'s.

Also, if the function φ_W defined in (2) is not equal to $-\infty$ on a neighborhood of $(-\infty, 2]$ and $\varphi'_W(\beta) > 0$, then it follows from [34,1,4] that $\tau_\mu = \varphi_{W_\beta}$ on the interval $J = \{q \leq 1 : \varphi^*_{W_\beta}(\varphi'_{W_\beta}(q)) \geq 0\}$ almost surely. This yields $\tau_{\mu,\beta} \equiv \varphi_W$ on the interval $J_\beta = \beta \cdot J$.

7.2 The validity of **C2**$(h_{\mu,\beta})$ *when* μ *is a Mandelbrot measure*

Let $\varphi_j = j^{-1/2}\log^2(j)$ for every $j \geq 1$ and let $(j_p)_{p\geq 1}$ be an increasing sequence such that $\lim_{p\to\infty} j_p^{-1}\log_2 C_{j_p} = \beta$ (recall (7)). Let $(n_p)_{p\geq 1}$ be the sequence of integers defined by $n_p = \inf\{k : k(\tau'_\mu(1) - \varphi_k) \leq \log_2 C_{j_p}\}$. We can choose the sequence $(\beta_j)_{j\geq 1}$ of Lemma 1 so that $(j_p+1)\beta_{j_p} \leq n_p(\tau'_\mu(1) - \varphi_{n_p})$. This last technical point is used at the end of the proof of Proposition 14.

It is shown in [10] that properties **(1)** and **(2b)** of Definition 5 are fulfilled $\mathbb{P}_\mu$-almost surely by μ with our choice of φ_j. Moreover, by our choice of $(\beta_j)_{j\geq 1}$ in Lemma 1 and $\{(u_n, l_n)\}$ in **C2**$(h_{\mu,\beta})$, property **(2a)** of Definition 5 is automatically fulfilled. So it remains to show that properties **(3)** and **(4)** of Definition 5 are satisfied $\mathbb{P}_\mu$-almost surely and $\mathbb{P}_S \otimes \mathbb{P}_\mu$-almost surely respectively.

Property **(3)** comes from the statistical self-similarity of μ: For $v \in \mathcal{A}^*$, let μ^v be the measure constructed on $[0,1]$ in the same way as μ is, but with the family of random vectors $\Big((W_0^v(w),\dots,W_{b-1}^v(w))\Big)_{w\in\mathcal{A}^*} = \Big((W_0(v\cdot w),\dots,W_{b-1}(v\cdot w))\Big)_{w\in\mathcal{A}^*}$ instead of $\Big((W_0 w),\dots,W_{b-1}(w))\Big)_{w\in\mathcal{A}^*}$. Let $|v|$ stand for the length of the word v and define $L_v = \Big[\sum_{k=1}^{|v|} v_k b^{-k}, b^{-|v|} + \sum_{k=1}^{|v|} v_k b^{-k}\Big]$. By construction, $\mathbb{P}_\mu$-almost surely, the restriction of the measure μ to L_v is equal to $W_{v_1}(\emptyset)W_{v_2}(v_1)\cdots W_{v_{|v|}}(v_1\cdots v_{|v|-1})\cdot\mu^v\circ f_{L_v}$ (the invertible function f_{L_v} is defined in Definition 5 **(3)**). Consequently, property **(3)** holds $\mathbb{P}_\mu$-almost surely with the choice $\mu^{L_v} = \mu^v \circ f_{L_v}$.

For $n \geq 1$ let

$$U_n^v = \left\{ t \in [0,1] : \left\{ \begin{array}{c} \forall\ j \geq n,\ \forall\ k,\ |k - k_{j,t}^b| \leq 1, \\ \mu^v\Big([kb^{-j}, (k+1)b^{-j}]\Big) \leq b^{-j(\tau'_\mu(1)-\varphi_j)} \end{array} \right. \right\}.$$

Then let

$$n_v = \inf\Big\{n \geq 1 :\ \mu^v(U_n^v) \geq \|\mu^v\|/2\Big\}.$$

It remains us to show that $\mathbb{P}_S \otimes \mathbb{P}_\mu$ almost surely, there exists a dense subset $\mathcal{D}$ of $(1, \infty)$ such that for every $\delta \in \mathcal{D}$, for μ-almost every $u \in [0,1]$, there exists a an increasing sequence of integers $(j_k(u))_{k\geq 1}$ such that for every $k \geq 1$ there exists $L_{v_k} \in B^\delta_{j_k(u)}(u)$ satisfying $\lim_{k\to\infty} \frac{|v_k|}{j_k(u)} = \delta$ and

$$n_{v_k} \leq |v_k| \,.\, \varphi_{|v_k|} \quad \text{and} \quad b^{-|v_k|\varphi_{|v_k|}} \leq \|\mu^{v_k}\|. \tag{40}$$

The function F is still defined by $F(t) = \mu([0,t])$. For every $w \in \mathcal{A}^{n_p}$, let $N_w(\omega_S, \omega_\mu)$ be the number of points of the Poisson point process S falling in $F(L_w) \times (2^{-(j_p+1)}, 2^{-j_p}]$. Conditionally on μ, the variable N_w is a Poisson variable with intensity $\mu(L_w)C_{j_p}$. Then, the orthogonal projection of $S \cap (F(L_w) \times (2^{-(j_p+1)}, 2^{-j_p}])$ onto $F(L_w)$ is equal to $\{\zeta_1, \ldots, \zeta_{N_w}\}$, where $(\zeta_i)_{i\geq 1}$ is a sequence of independent random variables (under $\mathbb{P}_S$), uniformly distributed in $F(L_w)$.

We set $\zeta_w = \zeta_1$ and $\widetilde{\zeta}_w = F^{-1}(\zeta_w)$. If $\delta > 1$, $v(\delta, \widetilde{\zeta}_w)$ stands for the word of generation $[\delta|w|] + 1$ such that $\widetilde{\zeta}_w \in L_{v(\delta, \widetilde{\zeta}_w)}$.

If $t \in [0,1)$ and $n \geq 1$, we denote by $w_n(t)$ the element w of $\mathcal{A}^n$ such that $t \in L_w$.

The validity of **(4)** is then a consequence of the following propositions.

Proposition 13 *Let $\delta > 1$. With $\mathbb{P}$-probability 1, for μ-almost every t, if p is large enough, then (40) holds with $v_k = v(\delta, \widetilde{\zeta}_{w_{n_p}(t)})$.*

Proposition 14 *With $\mathbb{P}$-probability 1, for μ-almost every t, there are infinitely many p's such that $N_{w_{n_p}(t)} \geq 1$, that is $\zeta_{w_{n_p}(t)}$ is a jump point of X.*

For $n \geq 1$ and $v \in \mathcal{A}^*$ let $R_n(v) = \mu^v\Big((U_n^v)^c\Big)$. The proof of Proposition 13 uses the following result which follows from our choice for φ_j and results in [10].

Lemma 7 *For every $n \geq 1$, the random variables $R_n(v)$, $v \in \mathcal{A}^*$, are identically distributed. Denote $R_n(\emptyset) = R_n$. Then, for all $h \in (0,1)$, $\mathbb{E}\Big((R_n)^h\Big) = O(b^{-\log^2(n)})$.*

PROOF. (of Proposition 13). Let $\mathcal{Q}$ be the probability measure defined on $\mathcal{B}(\Omega_S)\otimes\mathcal{B}(\Omega_\mu)\otimes\mathcal{B}([0,1])$ by

$$\mathcal{Q}(A)=\mathbb{E}\Big(\int_{[0,1]}\mathbf{1}_A(\omega_S,\omega_\mu,t)\,\mu(dt)\Big).$$

Notice that $\mathcal{Q}$-almost surely means for $\mathbb{P}_S\otimes\mathbb{P}_\mu$-almost every (ω_S,ω_μ), for μ_{ω_μ}-almost every t. Let $\psi_j=j\varphi_j$, $r_p=[\delta n_p]+1$ and $\rho_p=\log^{3/2}(n_p)$. By the Borel-Cantelli lemma, and since $\rho_p\leq\psi_{r_p}$ for p large enough, it is enough to prove that

$$\sum_{p\geq 1}\mathcal{Q}\left(b^{\rho_p}R_{\psi_{r_p}}\Big(v(\delta,\widetilde{\zeta}_{w_{n_p}(t)})\Big)\geq 1/2\right)<\infty \tag{41}$$

$$\sum_{p\geq 1}\mathcal{Q}\left(\|\mu^{v(\delta,\widetilde{\zeta}_{w_{n_p}(t)})}\|\leq b^{-\rho_p}\right)<\infty. \tag{42}$$

Let us establish (41). For $p\geq 1$ and $h\in(0,1)$, one has

$$\mathcal{Q}\left(b^{\rho_p}R_{\psi_{r_p}}\Big(v(\delta,\widetilde{\zeta}_{w_{n_p}(t)})\Big)\geq 1/2\right)\leq 2^h b^{\rho_p h}\mathbb{E}_{\mathcal{Q}}\left(R_{\psi_{r_p}}\Big(v(\delta,\widetilde{\zeta}_{w_{n_p}(t)})\Big)^h\right). \tag{43}$$

In addition, $\mathbb{E}_{\mathcal{Q}}\left(R_{\psi_{r_p}}\Big(v(\delta,\widetilde{\zeta}_{w_{n_p}(t)})\Big)^h\right)=\mathbb{E}\Big(\sum_{w\in\mathcal{A}^{n_p}}R_{\psi_{r_p}}\Big(v(\delta,\widetilde{\zeta}_w)\Big)^h\mu(L_w)\Big).$

Given $u,w\in\mathcal{A}^*$, $w\preceq u$ means that $L_u\subset L_w$. One has

$$\begin{aligned}
&\mathbb{E}\left(R_{\psi_{r_p}}\Big(v(\delta,\widetilde{\zeta}_w)\Big)^h\mu(L_w)\right)=\sum_{u\in\mathcal{A}^{[\delta n_p]+1},\ w\preceq u}\mathbb{E}\left(\mathbf{1}_{L_u}(\widetilde{\zeta}_w)R_{\psi_{r_p}}(u)\mu(L_w)\right)\\
&=\sum_{u\in\mathcal{A}^{[\delta n_p]+1},\ w\preceq u}\mathbb{E}\left(\mathbf{1}_{F(L_u)}(\zeta_w)R_{\psi_{r_p}}(u)^h\mu(L_w)\right)\\
&=\sum_{u\in\mathcal{A}^{[\delta n_p]+1},\ w\preceq u}\mathbb{E}_{\mathbb{P}_\mu}\left(\mathbb{P}_S\Big(\zeta_w\in F(L_u)\Big)R_{\psi_{r_p}}(u)^h\mu(L_w)\right)\\
&=\sum_{u\in\mathcal{A}^{[\delta n_p]+1},\ w\preceq u}\mathbb{E}\left(\frac{|F(L_u)|}{|F(L_w)|}R_{\psi_{r_p}}(u)^h\mu(L_w)\right)=\sum_{u\in\mathcal{A}^{[\delta n_p]+1},\ w\preceq u}\mathbb{E}\left(|F(L_u)|R_{\psi_{r_p}}(u)^h\right).
\end{aligned}$$

It follows from the previous equality and the structure of μ that

$$\mathbb{E}_{\mathcal{Q}}\Big(R_{\psi_{r_p}}\Big(v(\delta,\widetilde{\zeta}_{w_{n_p}(t)})\Big)^h\Big)=\mathbb{E}(R_{\psi_{r_p}}(u)^h\|\mu^u\|),$$

where u is any element of $\mathcal{A}^*$. Since it is assumed that μ is positive with probability 1 as well as $\mathbb{E}(\sum_{k=0}^{b-1}W_k^\alpha)<\infty$ for some $\alpha>1$, it follows from [15] that α can be chosen so that $\mathbb{E}(\|\mu\|^\alpha)<\infty$. Consequently, by using the

Hölder inequality one gets $\mathbb{E}(R_{\psi_{r_p}}(u)^h \|\mu^u\|) \leq \mathbb{E}(\|\mu\|^\alpha)^{1/\alpha} \mathbb{E}(R_{\psi_{r_p}}^{h\alpha'})^{1/\alpha'}$, where $\alpha^{-1} + \alpha'^{-1} = 1$. The conclusion follows by using (43) together with Lemma 7 applied with h small enough.

Let us move to (42). For $p \geq 1$ and $h \in (0,1)$ one has

$$\mathcal{Q}\left(\|\mu^{v(\delta,\widetilde{\zeta}_{w_{n_p}(t)})}\| \leq b^{-\rho_p}\right) \leq b^{-\rho_p h} \mathbb{E}_{\mathcal{Q}}\left(\|\mu^{v(\delta,\widetilde{\zeta}_{w_{n_p}(t)})}\|^{-h}\right).$$

Computations comparable to those used in establishing (41) show that

$$\mathbb{E}_{\mathcal{Q}}\left(\|\mu^{v(\delta,\widetilde{\zeta}_{w_{n_p}(t)})}\|^{-h}\right) = \mathbb{E}\left(\|\mu\|^{1-h}\right) < \infty.$$

The conclusion follows from our choice for ρ_p.

PROOF. of Proposition 14. Let $\omega_\mu \in \Omega_\mu$ such that $\mu = \mu(\omega_\mu)$ is defined and positive, and let $t \in (0,1)$ in the set of full μ-measure described in property **(2b)** of Definition 5. The random variables $N_{w_{n_p}(t)}(\cdot, \omega_\mu)$, $p \geq 1$, are $\mathbb{P}_S$ independent, and

$$\mathbb{P}_S\Big(N_{w_{n_p}(t)}(\cdot,\omega_\mu) \geq 1\Big) = 1 - \exp\Big(-\mu(L_{w_{n_p}(t)})C_{j_p}\Big).$$

Due to the definition of n_p and property **(2b)**, for p large enough one has $1 - \exp\Big(-\mu(L_{w_{n_p}(t)})C_{j_p}\Big) \geq 1 - \exp(-1)$, so $\sum_{p\geq 1} \mathbb{P}_S(N_{w_{n_p}(t)}(\cdot,\omega_\mu) \geq 1) = \infty$. The Borel-Cantelli lemma allows to conclude that $\mathbb{P}_S$-almost surely $N_{w_{n_p}(t)}(\omega_S, \omega_\mu) \geq 1$ for infinitely many p. Since this holds $\mathbb{P}_\mu$-almost surely, for μ-almost every t, we have the desired result by the Fubini theorem.

A final important remark is that the constraint $(j_p+1)\beta_{j_p} \leq n_p(\tau_\mu'(1) - \varphi_{n_p})$ imposed on β_{j_p} ensures that $t \in [u_n - l_n/2, u_n, +l_n/2]$ if u_n stands for $\widetilde{\zeta}_{w_{n_p}(t)}$.

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