A LOCALIZED JARNIK-BESICOVITCH THEOREM

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Abstract. Fundamental questions in Diophantine approximation are related to the Hausdorff dimension of sets of the form $\{x \in \mathbb{R} : \delta_x = \delta\}$, where $\delta \geq 1$ and δ_x is the Diophantine approximation exponent of an irrational number x. We go beyond the classical results by computing the Hausdorff dimension of the sets $\{x \in \mathbb{R} : \delta_x = f(x)\}$, where f is a continuous function. Our theorem applies to the study of the approximation exponents by various approximation families. It also applies to functions f which are continuous outside a set of prescribed Hausdorff dimension.

1. Introduction

Recall that the irrationality exponent δ_x of an irrational number $x \in \mathbb{R}$ is the supremum of those real numbers $\delta \geq 0$ for which the inequality

$$\left| x - \frac{p}{q} \right| \leq \frac{1}{q^{2\delta}} \tag{1}$$

is satisfied for infinitely many (irreducible) rational numbers p/q. From Dirichlet's theorem, it is known that $\delta_x \geq 1$ for all irrational numbers, and it is classical that Lebesgue-almost all real numbers have their irrationality exponent equal to 1. Determining the Hausdorff dimension of the level sets of the irrationality exponent δ is a historical issue. The famous Jarnik-Besicovitch theorem gives an answer to this problem [27, 28, 10]. For $\delta \geq 1$, let us introduce the sets

$$\mathcal{L}_\delta = \{x \in \mathbb{R} : \delta_x \geq \delta\} \quad \text{and} \quad \widetilde{\mathcal{L}}_\delta = \{x \in \mathbb{R} : \delta_x = \delta\}. \tag{2}$$

Theorem 1.1. *For every $\delta \geq 1$, $\dim_{\mathcal{H}} \mathcal{L}_\delta = \dim_{\mathcal{H}} \widetilde{\mathcal{L}}_\delta = 1/\delta$.*

Many works related to Theorem 1.1 have been achieved (see [29, 30, 12, 13], as well as the monograph [7] and references therein). In particular, similar Diophantine approximation problems have been considered in limit sets of groups or in Julia sets of rational maps [15, 37, 21, 22], in mathematical physics and dynamical systems when studying resonance problems [1, 32, 33, 34, 8] and when measuring the distribution of Hölder singularities of measures and functions [23, 26, 24, 19, 3, 4, 5].

The purpose of this paper is to study sets of real numbers for which the irrationality exponent is not fixed in advance, but it may vary with x in a continuous way. More precisely, we prove the following theorem.

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Theorem 1.2. *Let $f : \mathbb{R} \longrightarrow [1, +\infty)$ be a continuous function.*
Consider the sets

$$\mathcal{L}(f) = \{x \in \mathbb{R} : \ \delta_x \geq f(x)\} \ \ \text{and} \ \ \widetilde{\mathcal{L}}(f) = \{x \in \mathbb{R} : \ \delta_x = f(x)\}.$$

Then the sets $\mathcal{L}(f)$ and $\widetilde{\mathcal{L}}(f)$ are dense in $\mathbb{R}$, and we have

$$\dim_{\mathcal{H}} \mathcal{L}(f) \ = \ \dim_{\mathcal{H}} \widetilde{\mathcal{L}}(f) \ = \ \frac{1}{\min\{f(x) : x \in \mathbb{R}\}}. \tag{3}$$

Theorem 1.2 generalizes the Jarnik-Besicovitch Theorem 1.1. In view of Theorem 1.1, the common value for the Hausdorff dimensions of $\mathcal{L}(f)$ and $\widetilde{\mathcal{L}}(f)$ is the biggest value one could expect. The proof of Theorem 1.2 is based on a delicate analysis of the distribution of the rational numbers. Also, it is worth noting that, even if we compute the Hausdorff dimension of $\widetilde{\mathcal{L}}(f)$, our approach does not build explicit examples of irrational points x satisfying $\delta_x = f(x)$ (it seems non-trivial, though possible, to exhibit such points).

Theorem 1.2 makes it possible to answer the following questions:

Are there real numbers $x \in [0, 1]$ satisfying $\delta_x = 1 + x$? $\quad \delta_x = 1/x$?

This question is of course not reachable via Jarnik's result, for which the approximation exponent is a fixed number $\delta \geq 1$, independent of x. Theorem 1.2 implies for instance that:

- for every real numbers $0 < a < b < 1$,
$$\dim_{\mathcal{H}} \Big\{x \in [a, b] : \ \delta_x = 1 + x\Big\} = \frac{1}{1 + a},$$
- for every real numbers $0 < a < b < 1$, for every $\alpha \geq 1$,
$$\dim_{\mathcal{H}} \Big\{x \in [a, b] : \delta_x = \frac{\alpha}{x}\Big\} = \frac{b}{\alpha},$$
- and if $[a, b] \subset \Big[\frac{1}{6}, \frac{5}{6}\Big]$, then
$$\dim_{\mathcal{H}} \Big\{x \in [a, b] : \ \delta_x = 2\sin(\pi x)\Big\} \ = \ \frac{1}{\min\big(2\sin(\pi a), 2\sin(\pi b)\big)}.$$

In the above equalities, the dimensions depend on the range of x. This was expected, since the conditions we impose on x depend on the non-constant continuous function f.

It turns out that our approach to prove Theorem 1.2 makes it also possible to find the Hausdorff dimension of the level sets of approximation exponents associated with *systems of points* different from rational numbers. In the following, we restrict ourselves to $(0, 1)$, the extension to $\mathbb{R}$ is obvious.

Definition 1.3. *A system $\mathcal{S}$ is a sequence of couples $\big((x_n, r_n)\big)_{n\geq 1}$, where $(x_n)_{n\geq 1}$ is a sequence of real numbers of $(0, 1)$ and $(r_n)_{n\geq 1}$ is a non-increasing sequence of real numbers converging to 0 when n tends to infinity.*

The approximation exponent, still denoted by δ_x, of a real number $x \in [0,1]$ by the system $\mathcal{S}$ is the supremum of those real numbers $\delta \geq 0$ for which the inequality

$$|x - x_n| \leq (r_n)^\delta$$

is realized for infinitely many integers n. Then, as in (2), one defines $\mathcal{L}_\delta(\mathcal{S}) = \{x \in (0,1) : \delta_x \geq \delta\}$ and $\widetilde{\mathcal{L}}_\delta(\mathcal{S}) = \{x \in (0,1) : \delta_x = \delta\}$.

Notice that under the property

$$\limsup_{n\to\infty} B(x_n, r_n) = (0,1) \tag{4}$$

we have

$$\delta_x \geq 1 \quad \text{for all } x \in (0,1). \tag{5}$$

We can now state our main result. It invokes a property $\mathcal{P}$, detailed in Section 2.3, which is related to fine information regarding the distribution of the points $(x_n)_{n\geq 1}$. This property $\mathcal{P}$ is verified by the *rational* system $\mathcal{R} = \big((p/q, 1/q^2)\big)_{q\geq 1,\, 0<p\leq p-1}$, the *dyadic* system $\mathcal{D} = \big((k2^{-j}, 2^{-j})\big)_{j\geq 1, k\in\{1,\dots,2^j-1\}}$, the *inhomogeneous* system $\mathcal{I} = \Big(\big(\{n\alpha\}, \frac{1}{n}\big)\Big)_{n\geq 1}$ (for $\alpha \notin \mathbb{Q}$), as well as some random systems defined by Poisson point processes (Section 5).

Theorem 1.4. *Consider a system $\mathcal{S} = \big((x_n, r_n)\big)_{n\geq 1}$ as in Definition 1.3. Assume that (4) and property $\mathcal{P}$ hold for the system $\mathcal{S}$.*

Let $f : (0,1) \to [1, +\infty)$ be a continuous function. For $\Omega \subset (0,1)$ let

$$\mathcal{L}(\Omega, f) = \{x \in \Omega : \ \delta_x \geq f(x)\} \tag{6}$$

$$\widetilde{\mathcal{L}}(\Omega, f) = \{x \in \Omega : \ \delta_x = f(x)\}. \tag{7}$$

If Ω is a non-trivial subinterval of $(0,1)$, then the sets $\mathcal{L}(\Omega, f)$ and $\widetilde{\mathcal{L}}(\Omega, f)$ are dense in Ω and we have

$$\dim_{\mathcal{H}} \mathcal{L}(\Omega, f) = \dim_{\mathcal{H}} \widetilde{\mathcal{L}}(\Omega, f) = \frac{1}{\inf\{f(x) : x \in \Omega\}}. \tag{8}$$

Remark 1.5. *(1) Given a system $\mathcal{S}$ as in Definition 1.3, the authors of* [25, 15] *obtained the following* ubiquity *theorem, which yields an extension of Theorem 1.1 (concerning the lower bound of the sets $\mathcal{L}_\delta(\mathcal{S})$):*

Theorem 1.6. [25, 15] *Let ℓ stand for the one-dimensional Lebesgue measure. Let $\mathcal{S}$ be a system as in Definition 1.3. If*

$$\ell\Big(\mathcal{L}_1(\mathcal{S})\Big) = 1, \tag{9}$$

then for every $\delta \geq 1$, $\dim_{\mathcal{H}} \mathcal{L}_\delta(\mathcal{S}) \geq 1/\delta$.

(2) Observe that $\widetilde{\mathcal{L}}(\Omega, f)$ cannot be written as limsup set, and that the sets $\mathcal{L}(\Omega, f)$ and $\widetilde{\mathcal{L}}(\Omega, f)$ cannot be studied by Khintchine-like formulas or by mass transference formulas as stated in [7, 9] *(unless a localized version of these theories is developed). Moreover, they do not possess any large intersection properties* [18]*, due to the presence of the non-constant function f.*

We refer to Theorem 1.4 as a *localized Diophantine approximation* for the following reason. Usually, sets like $\widetilde{\mathcal{L}}_\delta(\mathcal{S})$ enjoy the following property: If Ω is a non-trivial subinterval of $(0,1)$, then $\dim_{\mathcal{H}}(\widetilde{\mathcal{L}}_\delta(\mathcal{S})\cap\Omega) = \dim_{\mathcal{H}} \widetilde{\mathcal{L}}_\delta(\mathcal{S})$. This is absolutely not the case for the sets $\mathcal{L}((0,1),f)$ and $\widetilde{\mathcal{L}}((0,1),f)$. Indeed, although these sets are dense in Ω, in general they are mostly localized around those elements of Ω at which f approaches its infimum. This induces that $\dim_{\mathcal{H}}(\widetilde{\mathcal{L}}(\Omega,f)) = \dim_{\mathcal{H}} \widetilde{\mathcal{L}}((0,1),f)$ only if f and its restriction to Ω have the same infimum. Hence, denoting $\dim_{\mathcal{H}} \widetilde{\mathcal{L}}((0,1),f)$ by s, in general there is no dimension function ϕ such that $\lim_{r\to 0^+} \log(\phi(r))/\log(r) = s$ and the associated Hausdorff measure of $\widetilde{\mathcal{L}}(\Omega,f)$ is positive. This is an important difference with the sets $\widetilde{\mathcal{L}}_\delta(\mathcal{S})$.

Let us illustrate our purpose with a concrete example.

In $(0,1)$, consider the system $\mathcal{R}$ associated with the rational numbers and the function $f(x) = 1+x$ (the crucial property is the strict monotonicity of f). We are interested in $\mathcal{L}((0,1),f) = \{x \in (0,1) : \delta_x \geq 1+x\}$ and $\widetilde{\mathcal{L}}((0,1),f) = \{x \in (0,1) : \delta_x = 1+x\}$.

Jarnik-Besicovich's theorem obviously implies that $\dim_{\mathcal{H}} \mathcal{L}((0,1),f) = 1$. Indeed, using that $1+x$ tends to 1 when $x>0$ tends to 0, for every $\varepsilon>0$, the set $\mathcal{L}((0,1),f)\cap[0,\varepsilon]$ contains all the real numbers whose approximation rate δ_x is larger than $1+\varepsilon$. These real numbers form a set of Hausdorff dimension $1/(1+\varepsilon)$. Letting ε tend to zero yields the result.

Similar arguments imply that for every $\varepsilon>0$, $\widetilde{\mathcal{L}}((0,1),f)\cap[\varepsilon,1]$ has Hausdorff dimension less than $1/(1+\varepsilon)<1$. However, Theorem 1.4 implies that the Hausdorff dimension of $\widetilde{\mathcal{L}}((0,1),f)$ is 1. Consequently, the elements of $\widetilde{\mathcal{L}}((0,1),f)$ responsible for the value 1 of the Hausdorff dimension of $\widetilde{\mathcal{L}}((0,1),f)$ are "localized" near 0. Observe that in this case, the infimum of f on $(0,1)$ is not reached.

The proof of Theorem 1.4 consists in constructing a family of Cantor sets $(\mathcal{K}_\varepsilon)_{\varepsilon>0}$, all included in $\widetilde{\mathcal{L}}(\Omega,f)$, which are located around points y such that $f(y)$ is closer and closer to the infimum of the function f. These Cantor sets will contain elements x with the desired approximation exponents $f(x)$. The sequence of dimensions $\dim_{\mathcal{H}} \mathcal{K}_\varepsilon$ will be increasing to $\dfrac{1}{\inf\{f(x) : x\in\Omega\}}$, as ε tends to zero.

We will also prove Theorem 1.7, which is determinant for its application to the analysis of the Hölder singularities of some Markov processes [2]. This extension addresses functions f which are continuous outside a set E with a given Hausdorff dimension. Its proof differs from the proof of Theorem 1.4 only by small technical details, this is explained in Section 3.7. We adopt the same notations as in Theorem 1.4.

Theorem 1.7. *Consider a system* $\mathcal{S} = \big((x_n, r_n)\big)_{n\geq 1}$ *as in Definition 1.3. Assume that (4) and property* $\mathcal{P}$ *hold for the system* $\mathcal{S}$*. Let* Ω *be a non-trivial subinterval of* $(0,1)$*,* $E \subset \Omega$*, and* $f : (0,1) \to [1,\infty)$ *a function whose restriction to* $\Omega \setminus E$ *is continuous.*

Suppose that $\dim_{\mathcal{H}} E < \dfrac{1}{\inf\{f(x) : x \in \Omega\setminus E\}}$*. Then*

$$\dim_{\mathcal{H}} \mathcal{L}(\Omega\setminus E, f) = \dim_{\mathcal{H}} \widetilde{\mathcal{L}}(\Omega\setminus E, f) = \frac{1}{\inf\{f(x) : x \in \Omega\setminus E\}}. \tag{10}$$

If, moreover, $\dim_{\mathcal{H}} E < \dfrac{1}{\sup\{f(x) : x \in \Omega\setminus E\}}$*, then the sets* $\mathcal{L}(\Omega\setminus E, f)$ *and* $\widetilde{\mathcal{L}}(\Omega\setminus E, f)$ *are dense in* Ω*.*

The paper is organized as follows. Property $\mathcal{P}$, as well as some preliminary results, are given in Section 2. The lower bound in the two-sided equality (10) is proved in Section 3, while the corresponding upper bound is demonstrated in Section 4. Finally, several examples of suitable systems (including the rational system) are studied in Section 5.

2. Definitions and property $\mathcal{P}$

2.1. **Recalls and notations.** We refer the reader to [17, 31] for the definition of the s-Hausdorff measures $\mathcal{H}^s$ and the Hausdorff dimension $\dim_{\mathcal{H}}$.

If m is a Borel probability measure over $[0,1]$, then its lower Hausdorff dimension is defined by

$$\dim_{\mathcal{H}*}(m) = \inf\{\dim_{\mathcal{H}} B : m(B) > 0\}.$$

For every integer $j \geq 0$, we denote by $\mathcal{G}_j$ the set of dyadic sub-intervals of $[0,1]$ of generation j, and $\mathcal{G}_*$ stands for $\bigcup_{j\geq 1} \mathcal{G}_j$. For any dyadic interval $I \in \mathcal{G}_*$, we set $g(I) = -\log_2(|I|)$, the dyadic generation of I (recall that $|I|$ stands for the diameter of I).

We denote by Φ the set of functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

- φ is a non-decreasing continuous functions such that $\varphi(0) = 0$,
- $r \mapsto r^{-\varphi(r)}$ is decreasing and tends to infinity as $x > 0$ tends to 0,
- for all real numbers $\alpha, \beta > 0$, the mapping $r \mapsto r^{\alpha - \beta\varphi(r)}$ is increasing in a neighborhood of 0.

For instance, a function like $r \mapsto \frac{1}{\log|\log r|}$ has the right behavior around 0 to belong to Φ.

We introduce the property $\mathcal{P}$ of a system $\mathcal{S}$ (as in Definition 1.3). $\mathcal{P}$ ensures an homogeneous repartition in $[0,1]$ of the points $(x_n)_{n\geq 1}$, and limits the overlaps between the intervals $B(x_n, r_n)$ with comparable lengths.

2.2. **Weak redundancy.**

Definition 2.1. *Given the system* $\mathcal{S}=\big((x_n,r_n)\big)_{n\geq 1}$*, we define the irreducible sub-system* $\big((y_n,\rho_n)\big)_{n\geq 1}$ *associated with* $\big((x_n,r_n)\big)_{n\geq 1}$ *as follows:*

$$\big((y_n,\rho_n)\big)_{n\geq 1}=\big((x_n,r_n)\big)_{n\geq 1,\ n=\min\{p\geq 1:\, x_p=x_n\}}.$$

If $x\in\{x_n:n\geq 1\}$, then the irreducible subsystem $\big((y_n,\rho_n)\big)_{n\geq 1}$ contains one (and only one) couple of the form (x,r), where $r=\max\{r_n:(x_n,r_n)\in\mathcal{S}\}$. This definition is needed since the initial system $\big((x_n,r_n)\big)_{n\geq 1}$ may be very redundant (this occurs when one x appears infinitely many times in the sequence $(x_n)_{n\geq 1}$, as in the rational system $\big((p/q,1/q^2)\big)_{q\geq 1,\,0\leq p\leq q-1}$).

Definition 2.2. *Let* $\big((x_n,r_n)\big)_{n\geq 1}$ *be a system, and consider its irreducible subsystem* $\big((y_n,\rho_n)\big)_{n\geq 1}$*. For any integer* $j\geq 0$*, we set*

$$\mathcal{T}_j=\big\{n:\quad 2^{-(j+1)}<\rho_n\leq 2^{-j}\big\}. \tag{11}$$

Definition 2.3. Weak redundancy: *The system* $\big((x_n,r_n)\big)_{n\geq 1}$ *is weakly redundant when there exists a sequence of integers* $(N_j)_{j\geq 0}$ *satisfying:*

1. $(N_j)_{j\geq 1}$ *is non-decreasing and* $\displaystyle\lim_{j\to\infty}\frac{\log_2 N_j}{j}=0$.
2. *for every* $j\geq 1$*,* $\mathcal{T}_j$ *can be decomposed into at most* N_j *pairwise disjoint subsets, say* $\mathcal{T}_{j,1},\dots,\mathcal{T}_{j,N_j}$*, such that for each* $1\leq i\leq N_j$*, the intervals* $(B(y_n,\rho_n))_{n\in\mathcal{T}_{j,i}}$ *are pairwise disjoint.*

By construction, for a weakly redundant system, each $\mathcal{T}_{j,i}$ has cardinality less than 2^{j+1}, and $\mathcal{T}_j$ has cardinality less than $N_j\cdot 2^{j+1}$. Hence, the weak redundancy ensures that every $t\in[0,1]$ is covered by at most N_j intervals of the form $B(y_n,\rho_n)$, $n\in\mathcal{T}_j$. The fact that N_j does not increase too fast toward infinity explains the appellation "weak redundancy" [4].

2.3. The fine non-overlapping property $\mathcal{P}$.

In order to obtain Theorems 1.4 and 1.7, the weak redundancy combined with the covering property (4) is not sufficient. An additional property is required on the system. We emphasize that $\mathcal{P}$, though technical, is satisfied by many natural systems, as explained in Section 5. It appears that, except for the random Poisson system, $\mathcal{P}$ and the weak redundancy are quite easy to check.

Definition 2.4. *Suppose that* $\mathcal{S}=\big((x_n,r_n)\big)_{n\geq 1}$ *is a weakly redundant system, and consider the sequence* $(N_j)_{j\geq 1}$ *associated with* $\mathcal{S}$ *by Definition 2.3.*

There exists a continuous function $\psi:\mathbb{R}_+\to\mathbb{R}_+$ *such that* $\psi(0)=0$ *and for every* $j\geq 1$*,* N_j *can be written as*

$$N_j=2^{j\psi(2^{-j})}. \tag{12}$$

For every $\varphi\in\Phi$ *and for every* $j\geq 1$*, we define*

$$\begin{aligned}\gamma(j) &= \max\left\{k\in\mathbb{N}:N_k2^k\leq 2^{-j\varphi(2^{-j})}2^j\right\}\\ &= \max\left\{k\in\mathbb{N}:2^{k(1+\psi(2^{-k}))}\leq 2^{j(1-\varphi(2^{-j}))}\right\}.\end{aligned} \tag{13}$$

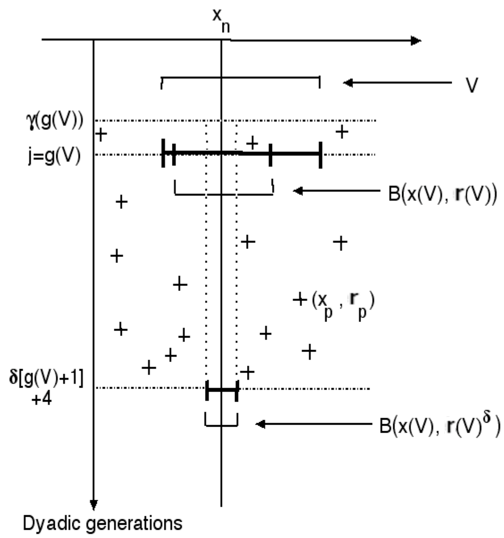

FIGURE 1. Property $\mathcal{P}(V,\delta)$

Obviously $\gamma(j)\le j$*, and we can write the difference* $j-\gamma(j)$ *as*

$$j-\gamma(j)=j\cdot\theta(2^{-j}), \tag{14}$$

for some continuous mapping $\theta:\mathbb{R}_+\to\mathbb{R}_+$ *such that* $\theta(0)=0$*.*

The sequence $(\gamma(j))_{j\ge1}$ and the mapping θ depend on the sequence $(N_j)_{j\ge1}$ and on φ. Nevertheless, in the following, we omit to write this dependence, since by Property $\mathcal{P}$, both $(N_j)_{j\ge1}$ and φ will be fixed once for all.

Definition 2.5. *Let* $\varphi\in\Phi$ *and* $(N_j)_{j\ge1}$ *be defined as in Definition 2.4. Let* $V\in\mathcal{G}_*$ *be a dyadic interval in* $[0,1]$*. Let* $\delta>1$ *be a real number.*

Recall that $g(V)=-\log_2|V|$ *is the dyadic generation of* V*.*

The property $\mathcal{P}(V,\delta)$ *is said to hold when there exists* $x(V)\in V\subset[0,1]$ *and a positive real number* $r(V)$ *satisfying:*

- $(x(V),r(V))\in\mathcal{S}$*,*
- $2^{-g(V)-1}\le r(V)<2^{-g(V)}$*,*
- *and*

$$B\big(x(V),r(V)^\delta\big)\bigcap\Big\{x_p:2^{-[\delta(g(V)+1)]+4}\le r_p<2^{-\gamma(g(V))}\Big\}=\{x(V)\}.$$

The notation $[y]$ stands for the integer part of a real number y. Recall that $\gamma\big(g(V)\big)\le g(V)$, and note that $[\delta g(V)]$ is heuristically the generation of the largest dyadic interval included in the contracted interval $B\left(x(V),r(V)^\delta\right)$.

$\mathcal{P}(V,\delta)$ holds when, except $x(V)$, all the elements x_p, where p ranges over the indices such that $\gamma\big(g(V)\big)\le-\log_2 r_p<[\delta(g(V)+1)]$ +4, avoids the contracted interval $B\left(x(V),r(V)^\delta\right)$ (see Figure 1). The constant 4 is due to technicalities along the proof. Note that $\mathcal{P}(V,\delta)$ depends on $(N_j)_{j\ge1}$ and φ

via γ (formula (13)), but as said above we do not mention this dependence since $(N_j)_{j\geq 1}$ and φ are going to be fixed by $\mathcal{P}$.

$\mathcal{P}(V,\delta)$ seems to be a reasonable property, but maybe not simultaneously for all dyadic intervals V and every δ. Property $\mathcal{P}$ is meant to ensure the validity of $\mathcal{P}(V,\delta)$ for a big enough set of intervals V and exponents δ.

Definition 2.6. Property $\mathcal{P}$: *A system $\mathcal{S}$ satisfies $\mathcal{P}$ when $\mathcal{S}$ is a weakly redundant system and when there exists :*

- *a function $\varphi \in \Phi$,*
- *a non-decreasing sequence of integers $(N_j)_{j\geq 1}$ as in Definition 2.4,*
- *a continuous function $\kappa : (1,+\infty) \to (0,1]$,*
- *a dense subset Δ of $(1,\infty)$,*

with the following property:

For every $\delta \in \Delta$, for every dyadic interval U of $[0,1]$, there are infinitely many integers $j \geq g(U)$ satisfying

$$\#\mathcal{Q}(U,j,\delta) \;\geq\; \kappa(\delta)\cdot 2^{j-g(U)}, \tag{15}$$

where

$$\mathcal{Q}(U,j,\delta) = \left\{V \in \mathcal{G}_j :\; V \subset U \quad \text{and} \quad \mathcal{P}\big(V,\delta\big) \text{ holds}\right\}.$$

Observe that $2^{j-g(U)}$ is the number of dyadic intervals V of generation $j \geq g(U)$ included in U. The set $\mathcal{Q}(U,j,\delta)$ contains those among these intervals V enjoying the property $\mathcal{P}\big(V,\delta\big)$. As claimed above, $\mathcal{P}$ guarantees that given a dyadic interval U and $\delta \in \Delta$, infinitely often a given proportion of the dyadic subintervals V of generation j included in U satisfies $\mathcal{P}\big(V,\delta\big)$.

Remark 2.7. *For the deterministic systems of Section 5, the function κ can be taken constant (i.e. $\kappa(\delta) = \kappa \in (0,1)$), and Δ is $(1,\infty)$. The dependence in δ of κ and the fact that Δ may be countable and dense in $(1,\infty)$ are introduced to include the Poisson point processes in the upper-half plane.*

In the following, a system $\mathcal{S}$ will always satisfy $\mathcal{P}$. Hence, φ and $(N_j)_{j\geq 1}$ are given, and all the parameters introduced from now on depend on them.

2.4. **A preliminary result.**

We shall need the following lemma, which requires only the weak redundancy property.

Lemma 2.8. *Let $\big((x_n,r_n)\big)_{n\geq 1}$ be a weakly redundant system in $[0,1]$ and let $\big((y_n,\rho_n)\big)_{n\geq 1}$ be its irreducible subsystem (we adopt the notations of Definition 2.3).*

For every $\delta > 1$, every dyadic interval $U \in \mathcal{G}_*$, and every integer $j \geq \delta\cdot g(U)$, let us introduce the set of dyadic intervals of generation j:

$$\widetilde{\mathcal{Q}}(U,j,\delta) = \Big\{V \in \mathcal{G}_j : V \subset U \text{ and } V \bigcap \Big(\bigcup_{k=g(U)}^{\gamma(j)} \bigcup_{p\in\mathcal{T}_k} B(y_p,(\rho_p)^{\delta})\Big) \neq \emptyset \Big\}.$$

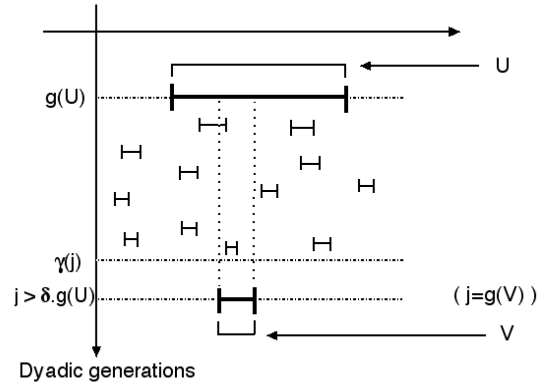

FIGURE 2. An illustration of $V \in \widetilde{\mathcal{Q}}(U,j,\delta)$

Then, there exists a universal constant C such that

$$\#\widetilde{\mathcal{Q}}(U,j,\delta) \leq C \cdot 2^{j-g(U)} \left[2^{-j\varphi(2^{-j})} + \sum_{g(U)\leq k\leq j/\delta} 2^{-k(\delta-1-\psi(2^{-k}))}\right]. \tag{16}$$

The set $\widetilde{\mathcal{Q}}(U,j,\delta)$ contains the dyadic intervals of generation j included in U which intersect the "irreducible" intervals $B(y_p,\rho_p)$ when p ranges in $\mathcal{T}_k$, for all indices $k \in \{g(U),...,\gamma(j)\}$. It is crucial in the construction of Cantor sets that $\widetilde{\mathcal{Q}}(U,j,\delta)$ does not contain too many elements (see Figure 2).

Proof of Lemma 2.8. Let $U \in \mathcal{G}_*$ and $j \geq \delta \cdot g(U)$. Let k be an integer such that $k \in \{g(U),...,\gamma(j)\}$. We are going to count the number of dyadic intervals V in $\mathcal{G}_j$ included in U which intersect intervals of the form $B(y_p,(\rho_p)^\delta)$ for some $p \in \mathcal{T}_k$. Two cases shall be distinguished:

• $g(U) \leq k \leq j/\delta$**:** When $p \in \mathcal{T}_k$, $|B(y_p,(\rho_p)^\delta)| \leq 2^{1-k\delta}$. Hence, $B(y_p,(\rho_p)^\delta)$ intersects at most $2 \cdot 2^j 2^{1-k\delta} + 2 \leq C \cdot 2^j 2^{-k\delta}$ dyadic intervals of $\mathcal{G}_j$, for some universal constant C.

Moreover, by construction, $\mathcal{T}_k = \bigcup_{1\leq l\leq N_k} \mathcal{T}_{k,l}$, where for each $\mathcal{T}_{k,l}$, the intervals $B(y_n,\rho_n)$, $n \in \mathcal{T}_{k,l}$, are pairwise disjoint closed intervals of length larger than $2.2^{-(k+1)}$. Consequently, for $1 \leq l \leq N_k$, the cardinality of those integers $p \in \mathcal{T}_{k,l}$ satisfying $B(y_p,\rho_p) \cap U \neq \emptyset$ is at most $C \cdot 2^{k-g(U)}$, again for some constant C. Thus, there are at most $C \cdot N_k 2^{k-g(U)}$ integers $p \in \mathcal{T}_k$ satisfying $B(y_p,\rho_p) \cap U \neq \emptyset$.

Combining these remarks, the number of dyadic intervals of $\mathcal{G}_j$ included in U which meet an interval $B(y_p,(\rho_p)^\delta)$, for some $p \in \mathcal{T}_k$, is less than

$$CN_k 2^{k-g(U)} \cdot 2^j 2^{k\delta} \quad = \quad C \cdot 2^{j-g(U)} N_k 2^{-k(\delta-1)}.$$

• $j/\delta < k \leq \gamma(j)$**:** for every $p \in \mathcal{T}_k$, $|B(y_p,(\rho_p)^\delta)| \leq 2^{1-j}$, thus $B(y_p,(\rho_p)^\delta)$ intersects at most 3 elements of $\mathcal{G}_j$. Consequently, the cardinality of the subset of $\mathcal{G}_j$ whose elements are included in U and meet an interval $B(y_p,(\rho_p)^\delta)$ for some $p \in \mathcal{T}_k$ is at most $3 \cdot N_k 2^{k-g(U)} = C \cdot N_k 2^{k-g(U)}$.

Summarizing the above estimates, we obtain

$$\#\widetilde{\mathcal{Q}}(U,j,\delta) \leq C \cdot \Big[2^{j-g(U)} \sum_{g(U)\leq k\leq j/\delta} N_k 2^{-k(\delta-1)} + 2^{-g(U)} \sum_{j/\delta<k\leq\gamma(j)} N_k 2^k\Big].$$

Using that $(N_k)_{k\geq 1}$ is non-decreasing, we get

$$\begin{aligned} 2^{-g(U)} \sum_{j/\delta<k\leq\gamma(j)} N_k 2^k \ &\leq\ 2^{-g(U)} N_{\gamma(j)} \sum_{j/\delta<k\leq\gamma(j)} 2^k \leq C \cdot N_{\gamma(j)}\, 2^{\gamma(j)}\, 2^{-g(U)} \\ &\leq\ C \cdot 2^{j-g(U)} 2^{-j\varphi(2^{-j})}, \end{aligned}$$

where we used the definition (13) of $\gamma(j)$ in the last inequality. Moreover, using the definition (12) of $\psi(2^{-k})$ based on N_k, we find that

$$2^{j-g(U)} N_k 2^{-k(\delta-1)} \leq 2^{j-g(U)} 2^{-k(\delta-1-\psi(2^{-k}))}.$$

Equation (16) follows easily.

3. Lower bound for the Hausdorff dimensions in Theorem 1.7

Until Section 3.7 we assume that the function f is continuous and that

$$h := \frac{1}{\inf\{f(x) : x \in \Omega\}} < 1. \tag{17}$$

We aim to show that $\widetilde{\mathcal{L}}(\Omega, f) = \{x \in \Omega : \delta_x = f(x)\}$ has Hausdorff dimension h. We are going to construct a family of Cantor sets included in $\widetilde{\mathcal{L}}(\Omega, f)$ and such that the supremum of their Hausdorff dimensions is h. We explain in Section 3.7 how to adapt the proof to the case where f is continuous except on a set E of Hausdorff dimension less than $\inf\{f(x) : x \in \Omega \setminus E\}$.

3.1. **Preliminary work.** Fix $\varepsilon \in (0, h)$. By definition (17) of h, there exists $y_\varepsilon \in \overset{\circ}{\Omega}$ such that $h - \varepsilon/2 \leq (f(y_\varepsilon))^{-1} \leq h$. Hence, using the continuity of f at y_ε, for every y in a neighborhood $\Omega_\varepsilon \subset \overset{\circ}{\Omega}$ small enough around y_ε, we get $h - \varepsilon \leq (f(y))^{-1} \leq h$. Equivalently, when ε is small enough, we have

$$\forall\, y \in \Omega_\varepsilon,\quad \frac{1}{h} \leq f(y) \leq \frac{1}{h-\varepsilon} \leq \frac{1}{h}\left(1 + 2\frac{\varepsilon}{h}\right). \tag{18}$$

Recall that property $\mathcal{P}$ provides us with two functions ϕ (12) and ψ, and with a set of admissible approximation exponents Δ. In every dyadic interval $V \in \mathcal{G}_*$ included in Ω_ε, we pick up an element $y_V \in V$ and we choose a real number $\delta(V) \in \Delta$ such that

$$\delta(V) \in \big[f(y_V) + (\varphi(|V|) + \psi(|V|)), f(y_V) + 3(\varphi(|V|) + \psi(|V|))\big]. \tag{19}$$

Observe that the real numbers $\delta(V)$ are bounded from above and below, since φ, ψ and f are continuous on Ω_ε. Moreover, by (18) there is a constant $\alpha > 1$ such that for every V having diameter small enough, one has

$$\delta(V) - 3(\varphi(|V|) + \psi(|V|)) \geq \alpha > 1. \tag{20}$$

Since the function $\kappa(\cdot)$ determined by property $\mathcal{P}$ is continuous, there is a constant $\kappa \in (0,1)$ such that for every δ belonging to the set $\{\delta(V) : V$ dyadic interval $\subset \Omega_\varepsilon\}$, for every dyadic interval $U \subset \Omega_\varepsilon$, (15) holds infinitely often with the same constant κ (instead of $\kappa(\delta)$). We choose κ as a power of 2, i.e. $\kappa = 2^{-K}$ for some constant $K \geq 3$. This will simplify a little bit the forthcoming constructions.

We now start the simultaneous construction of a Cantor set $\mathscr{K}_\varepsilon$ and a probability measure μ_ε supported by $\mathscr{K}_\varepsilon$ such that $\mathscr{K}_\varepsilon \subset \widetilde{\mathcal{L}}(\Omega, f) \cap \Omega_\varepsilon$ and $\dim_{\mathcal{H}*}(\mu_\varepsilon) \geq h_\varepsilon$, for some real number h_ε which satisfies $\lim_{\varepsilon\to 0^+} h_\varepsilon = h$.

Assume for a while that the construction is achieved. We deduce that

$$\dim_{\mathcal{H}} \widetilde{\mathcal{L}}(\Omega, f) \geq \dim_{\mathcal{H}} \mathscr{K}_\varepsilon \geq \dim_{\mathcal{H}*}(\mu_\varepsilon) \geq h_\varepsilon.$$

Letting ε tend to 0 yields the lower bound in Theorem 1.7, i.e.

$$\dim_{\mathcal{H}} \widetilde{\mathcal{L}}(\Omega, f) \geq h.$$

The Cantor set $\mathscr{K}_\varepsilon$ will be obtained as a limsup set of the form

$$\mathscr{K}_\varepsilon = \bigcap_{n\geq 0} \bigcup_{U\in\mathcal{F}_n} U,$$

where for every $n \geq 0$, $\mathcal{F}_n$ is a collection of pairwise disjoint closed dyadic intervals U such that each element of $\mathcal{F}_{n+1}$ is included in one (and by construction only one) element of $\mathcal{F}_n$.

The sequence $(\mathcal{F}_n)_{n\geq 0}$ is built by induction, as follows.

At first, we choose a dyadic interval U_0 included in Ω_ε, small enough so that $3(\varphi(|U_0|) + \psi(|U_0|)) \leq \varepsilon$ and $\dfrac{4}{\kappa} \leq |U_0|^{-\varphi(|U_0|)}$, where we recall that κ is the constant appearing in (15) (the dependence on δ has been removed by an argument above). We define $\mathcal{F}_0 = \{U_0\}$. This choice also implies that for any dyadic interval $U \subset U_0 \subset \Omega_\varepsilon$ that we are going to consider, we have (using (18) and (19))

$$\delta(U) \leq f(y_\varepsilon) + \varepsilon\left(2\frac{1}{h^2} + 1\right) =: H_\varepsilon. \tag{21}$$

Let

$$C_\varepsilon = 2^{-H_\varepsilon}/16. \tag{22}$$

3.2. **Construction of the first generation of the Cantor set, $\mathcal{F}_1$.** We use property $\mathcal{P}$ and Lemma 2.8 with $U = U_0$ and $\delta = \delta(U_0)$. This yields infinitely many integers $j \geq g(U_0)$ such that

$$\begin{aligned} \#\mathcal{Q}(U_0, j, \delta(U_0)) &\geq \kappa \cdot 2^{j-g(U_0)} \\ \text{and} \quad \#\widetilde{\mathcal{Q}}(U_0, j, \delta(U_0)) &\leq C \cdot 2^{j-g(U_0)}\Big(2^{-j\varphi(2^{-j})} + \\ &\qquad \sum_{g(U_0)\leq k\leq j/\delta(U_0)} 2^{-k(\delta(U_0)-1-\psi(2^{-k}))}\Big). \end{aligned}$$

We use (20) to bound from above the sum in the second equation:

$$\sum_{g(U_0)\leq k\leq j/\delta(U_0)} 2^{-k(\delta(U_0)-1-\psi(2^{-k}))} \quad\leq\quad \sum_{g(U_0)\leq k} 2^{-(\alpha-1)k} \leq C\cdot 2^{-(\alpha-1)g(U_0)}.$$

The constant C does not depend on U_0. Consequently, the second upper bound above can be simplified into

$$\#\widetilde{\mathcal{Q}}(U_0,j,\delta(U_0)) \quad\leq\quad C\cdot 2^{j-g(U_0)}(2^{-j\varphi(2^{-j})}+2^{-(\alpha-1)g(U_0)}).$$

Provided that U_0 has diameter small enough and j is large enough, we have $\#\widetilde{\mathcal{Q}}(U_0,j,\delta(U_0))\leq\kappa/4\cdot 2^{j-g(U_0)}$. From the inequalities between the cardinalities of $\mathcal{Q}(U_0,j,\delta(U_0))$ and $\widetilde{\mathcal{Q}}(U_0,j,\delta(U_0))$, we deduce that for j large enough, the set $\widetilde{\mathcal{F}}_1:=\mathcal{Q}(U_0,j,\delta(U_0))\setminus\widetilde{\mathcal{Q}}(U_0,j,\delta(U_0))$ contains at least $\kappa/2\cdot 2^{j-g(U_0)}$ elements. Moreover, we can find at least $\#\widetilde{\mathcal{F}}_1/2$ elements of $\widetilde{\mathcal{F}}_1$ which are distant from each other by at least 2^{-j}. Consequently, we can assume that there are exactly $\kappa/4\cdot 2^{j-g(U_0)}$ dyadic intervals in $\widetilde{\mathcal{F}}_1$, whose mutual distance is at least 2^{-j}.

By construction, each interval $\widetilde{V}\in\widetilde{\mathcal{F}}_1$ has the following characteristics:

(i) $\widetilde{V}\in\mathcal{G}_j$ (i.e. $j=g(\widetilde{V})$), $\widetilde{V}\subset U_0$,

(ii) $\widetilde{V}$ satisfies

$$\widetilde{V}\bigcap\Big(\bigcup_{k=g(U_0)}^{\gamma(j)}\ \bigcup_{p\in\mathcal{T}_k} B(y_p,(\rho_p)^{\delta(U_0)})\Big)=\emptyset.$$

(iii) $\mathcal{P}\big(\widetilde{V},\delta(U_0)\big)$ holds: $\widetilde{V}$ contains a real number $x(\widetilde{V})$ such that the couple $(x(\widetilde{V}),r(\widetilde{V}))$ belongs to $\mathcal{S}$ for some $r(\widetilde{V})$ with $2^{-j-1}\leq r(\widetilde{V})<2^{-j}$, and

$$B\left(x(\widetilde{V}),r(\widetilde{V})^{\delta(U_0)}\right)\bigcap\Big\{x_p:\gamma(j)\leq-\log_2 r_p<(j+1)\delta(U_0)+4\Big\}=\{x(\widetilde{V})\}.$$

Before going further, we need a new definition.

Definition 3.1. *For $\delta>1$, $\varepsilon>0$, $x\in\mathbb{R}$ and $r>0$ we define the annulus*

$$A(x,r,\delta,\varepsilon)=B(x,r^{\delta})\setminus B(x,r^{\delta+\varepsilon}).$$

For each $\widetilde{V}\in\widetilde{\mathcal{F}}_1$, consider the associated annulus

$$A(\widetilde{V})=A(x(\widetilde{V}),r(\widetilde{V}),\delta(U_0),\varphi(2^{-j})).$$

Remark 3.2. *The diameter of $A(\widetilde{V})$ is $2\cdot r(\widetilde{V})^{\delta(U_0)}=2^{1+\log_2(r(\widetilde{V})^{\delta(U_0)})}$. Provided that j is large, the "hole" in the annulus $A(\widetilde{V})$ is extremely small, since the ratio $\frac{r(\widetilde{V})^{\delta(U_0)}}{r(\widetilde{V})^{\delta(U_0)+\varphi(2^{-j})}}=r(\widetilde{V})^{-\varphi(2^{-j})}$ tends to infinity when j tends to infinity (recall that φ belong to the set of functions Φ and $r(\widetilde{V})\sim 2^{-j}$).*

Let $\widetilde{\widetilde{V}}$ be one of the largest closed dyadic intervals included in $A(\widetilde{V})\bigcap \widetilde{V}$. Using Remark 3.2, the generation $g(\widetilde{\widetilde{V}})$ of the dyadic interval $\widetilde{\widetilde{V}}$ is at most equal to $[-\log_2(r(\widetilde{V})^{\delta(U_0)})]+3$.

Then, *we choose the dyadic interval* V *as the dyadic subinterval of* $\widetilde{\widetilde{V}}$ *of generation* $g(\widetilde{\widetilde{V}})+1$ *which is the closest to* $x(\widetilde{V})$. We obtain that:

- the dyadic generation of V satisfies

$$[-\log_2(r(\widetilde{V})^{\delta(U_0)})] \le g(V) \le [-\log_2(r(\widetilde{V})^{\delta(U_0)})]+4, \tag{23}$$

- for each p such that $\gamma(j) \le -\log_2 r_p \le [(j+1)\delta(U_0)]+4$, if $x_p \neq x_n = x(\widetilde{V})$ then $x_p \notin B(x(\widetilde{V}), r(\widetilde{V})^{\delta(U_0)})$,
- for each p such that $\gamma(j) \le -\log_2 r_p \le [(j+1)\delta(U_0)]+4$, for all $x\in V$ we have (using the function θ defined by (14) in Definition 2.4)

$$\begin{aligned} |x-x_p| \ge |V| &\ge r(\widetilde{V})^{\delta(U_0)}/16 = 2^{(\gamma(j)-j)\delta(U_0)}\cdot 2^{-\gamma(j)\delta(U_0)}/16 \\ &\ge 2^{-j\theta(2^{-j})\delta(U_0)}\cdot r_p^{\delta(U_0)}/16 \ge r_p^{\delta(U_0)+\theta(2^{-j})\delta(U_0)/2}/16. \end{aligned}$$

  The last inequality follows from the fact that, when j is large enough, $r_p \le 2^{-\gamma(j)} \le 2^{-j/2}$. Finally, using (21), we see that

$$|x-x_p| \ge r_p^{\delta(U_0)+\theta(2^{-j})H_\varepsilon/2}/16. \tag{24}$$

When two dyadic intervals V and $\widetilde{V}$ are related via such a relationship, we say that V *is the contracted descendant of* $\widetilde{V}$.

The previous construction guarantees that (recall that $j=g(\widetilde{V})$):

- (23) holds,
- since $V\subset A(\widetilde{V})$, every element $x\in V$ satisfies

$$r(\widetilde{V}))^{\delta(U_0)+\varphi(2^{-j})} \le |x-x(\widetilde{V})| \le r(\widetilde{V}))^{\delta(U_0)}, \tag{25}$$

  i.e. x is approximated at rate $\in[\delta(U_0),\delta(U_0)+\varphi(2^{-j})]$ by $(x(\widetilde{V}), r(\widetilde{V}))$,
- for every $k\in\{g(U_0),...,\gamma(j)\}$, for every $p\in\mathcal{T}_k$,

$$V \bigcap B(x_p,(r_p)^{\delta(U_0)})=\emptyset,$$

  i.e. every $x\in V$ is not approximated at rate larger than $\delta(U_0)$ by these couples $(x_p,r_p)\in\mathcal{S}$,
- Combining the last item with (24), if an integer p is such that $2^{-g(V)}\le r_p \le 2^{-g(U_0)}$ and $x_p\neq x(\widetilde{V})$, then

$$V \bigcap B\big(x_p, r_p^{\delta(U_0)+\theta(2^{-j})H_\varepsilon/2}/16\big)=\emptyset.$$

By construction, we have $|V|\ge 2^{-(j+1)\delta(U_0)}/16$. Consequently, using (22), without loss of generality one can suppose that j is so large that

$$\text{for every } \widetilde{V}\in\mathcal{Q}(U_0,j,\delta(U_0)),\ \ |V|\ge C_\varepsilon\cdot 2^{-j\delta(U_0)}. \tag{26}$$

This yields a precise relationship between the diameter of a interval $\widetilde{V} \in \widetilde{\mathcal{F}}_1$ and the diameter of its contracted descendant V.

Since the properties described above occur for an infinite number of generations j, we choose j so large that

$$j \geq 2g(U_0) \quad \text{and} \quad \max\left(C_\varepsilon^{-1/\delta(U_0)}, 4/\kappa, 2^{g(U_0)}\right) \leq 2^{j\varphi(2^{-j})}. \tag{27}$$

This ensures that $4/\kappa \leq |V|^{-\varphi(|V|)}$, which will play a role in Section 3.5.

Now, consider the set $\mathcal{F}_1$ of contracted descendants of the elements of $\widetilde{\mathcal{F}}_1$:

$$\mathcal{F}_1 = \{V : V \text{ is the contracted descendant of some } \widetilde{V} \in \widetilde{\mathcal{F}}_1\}.$$

We construct a measure μ_ε on the algebra $\sigma_1 = \sigma(V : V \in \mathcal{F}_1)$ generated by the dyadic intervals of $\mathcal{F}_1$ by imposing:

$$\forall\, V \in \mathcal{F}_1, \quad \mu_\varepsilon(V) = (\#\mathcal{F}_1)^{-1}.$$

Let $V \in \mathcal{F}_1$. Recalling that $\#\widetilde{\mathcal{F}}_1 = \frac{\kappa}{4} \cdot 2^{j-g(U_0)}$, using (26) we get

$$\mu_\varepsilon(V) \leq \frac{4}{\kappa} \cdot 2^{g(U_0)-j} \leq \frac{4}{\kappa} \cdot 2^{g(U_0)} C_\varepsilon^{-1/\delta(U_0)} |V|^{1/\delta(U_0)}.$$

Using (27) we find

$$\mu_\varepsilon(V) \leq 2^{3j\varphi(2^{-j})} |V|^{1/\delta(U_0)}.$$

Due to the monotonicity of $r^{-\varphi(r)}$, we have $2^{3j\varphi(2^{-j})} \leq |V|^{-3\varphi(|V|)}$, when j is large. Hence,

$$\forall\ V \in \mathcal{F}_1, \quad \mu_\varepsilon(V) \leq |V|^{1/\delta(U_0) - 3\varphi(|V|)} \tag{28}$$

We now fix the integer $j = j_0$ so that all the properties above are satisfied. The last property of these intervals of first generation is that for every $V \neq V' \in \mathcal{F}_1$, the distance between V and V' is greater than 2^{-j_0}.

3.3. **Construction of $\mathcal{F}_2$, the second generation of the Cantor set.** Thanks to (27), for each interval $U_1 \in \mathcal{F}_1$, we have $4/\kappa \leq |U_1|^{-\varphi(|U_1|)}$.

First we work in a given $U_1 \in \mathcal{F}_1$. There are infinitely many integers $j \geq g(U_1)$ such that

$$\begin{aligned} \#\mathcal{Q}(U_1, j, \delta(U_1)) &\geq \kappa \cdot 2^{j-g(U_1)} \\ \#\widetilde{\mathcal{Q}}(U_1, j, \delta(U_1)) &\leq C \cdot 2^{j-g(U_1)} \Big(2^{-j\varphi(2^{-j})} \\ &\quad + \sum_{g(U_1) \leq k \leq j/\delta(U_1)} 2^{-k(\delta(U_1)-1-\psi(2^{-k}))}\Big). \end{aligned}$$

The arguments used in the first step to find an upper bound for the sum in the second inequality above also apply here: For large j, there is a subset $\widetilde{\mathcal{F}}_2(U_1)$ of cardinality $\dfrac{\kappa}{4} 2^{j-g(U_1)}$ in $\mathcal{Q}(U_1, j, \delta(U_1)) \setminus \widetilde{\mathcal{Q}}(U_1, j, \delta(U_1))$ such that

- the dyadic intervals $\widetilde{V}$ belonging to $\widetilde{\mathcal{F}}_2(U_1)$ are mutually distant from at least 2^{-j},

- each dyadic interval $\widetilde{V} \in \widetilde{\mathcal{F}}_2(U_1)$ satisfies simultaneously $\widetilde{V} \in \mathcal{G}_j$, $\widetilde{V} \subset U_1$, $\mathcal{P}\big(\widetilde{V}, \delta(U_1)\big)$ and

$$\widetilde{V} \bigcap \Big(\bigcup_{k=g(U_1)}^{\gamma(j)} \bigcup_{p \in \mathcal{T}_k} B(y_p, (\rho_p)^{\delta(U_1)}) \Big) = \emptyset.$$

As in the first step, we associate with every $\widetilde{V} \in \widetilde{\mathcal{F}}_2(U_1)$ a dyadic interval called its contracted descendant V, which enjoys the following properties:

- There exists an element $x(\widetilde{V}) \in \widetilde{V}$ and a positive real number $r(\widetilde{V})$ such that $(x(\widetilde{V}), r(\widetilde{V})) \in \mathcal{S}$, $r(\widetilde{V})$ satisfies $2^{-j-1} \leq r(\widetilde{V}) \leq 2^{-j}$, and every $x \in V$ is approximated at a rate belonging to $[\delta(U_1), \delta(U_1) + \varphi(2^{-j})]$ by $(x(\widetilde{V}), r(\widetilde{V}))$ (in the same sense as in (25)),
- if p is such that $2^{-g(V)} \leq r_p \leq 2^{-g(U_1)}$ and $x_p \neq x_n$, then

$$V \bigcap B\big(x_p, (r_p)^{\delta(U_1)+\theta(2^{-j})H_\varepsilon/2}/16\big) = \emptyset.$$

By construction,

$$\text{for every } \widetilde{V} \in \widetilde{\mathcal{F}}_2(U_1), \quad |V| \geq C_\varepsilon 2^{-j(U_1)\delta(U_1)}. \tag{29}$$

We now fix the integer $j = j(U_1)$ so that all the properties above are satisfied, and we set

$$\begin{aligned} \mathcal{F}_2(U_1) &= \{V : V \text{ is the contracted descendant of one } \widetilde{V} \in \widetilde{\mathcal{F}}_2(U_1)\}, \\ \text{and} \quad \mathcal{F}_2 &= \{V \in \mathcal{G}_* : \ \exists U_1 \in \mathcal{F}_1 \text{ such that } V \in \mathcal{F}_2(U_1)\}. \end{aligned}$$

The measure μ_ε can be extended into a Borel probability measure on the algebra $\sigma_2 = \sigma(L : L \in \mathcal{F}_1 \cup \mathcal{F}_2)$ by imposing

$$\text{for every } U_1 \in \mathcal{F}_1, \text{ for every } V \in \mathcal{F}_2(U_1), \quad \mu_\varepsilon(V) = \frac{\mu_\varepsilon(U_1)}{\#\mathcal{F}_2(U_1)}.$$

We choose $j_1 := \min(j(U_1) : U_1 \in \mathcal{F}_1)$ large enough so that for every $U_1 \in \mathcal{F}_1$, $j_1 \geq 2g(U_1)$ and

$$\max\Big(C_\varepsilon^{-1/\delta(U_1)}, 4/\kappa, 2^{g(U_1)}, |U_1|^{1/\delta(U_0)-3\varphi(|U_1|)}\Big) \leq 2^{j_1\varphi(2^{-j_1})}. \tag{30}$$

In particular, for every $V \in \mathcal{F}_2(U_1)$ we have $4/\kappa \leq |V|^{-\varphi(|V|)}$.

Let us check the scaling properties of the measure μ_ε on the elements of σ_2. Let $U_1 \in \mathcal{F}_1$ and $V \in \mathcal{F}_2(U_1)$. Combining (28) with (29) and the fact that $\#\widetilde{\mathcal{F}}_2(U_1) = \frac{\kappa}{4} \cdot 2^{j(U_1)-g(U_1)}$, we obtain that

$$\begin{aligned} \mu_\varepsilon(V) &= \frac{4}{\kappa} \cdot 2^{g(U_1)-j(U_1)} \mu_\varepsilon(U_1) \ \leq \ \frac{4}{\kappa} \cdot 2^{g(U_1)-j(U_1)} |U_1|^{1/\delta(U_0)-3\varphi(|U_1|)} \\ &\leq \frac{4}{\kappa} \cdot 2^{g(U_1)} |U_1|^{1/\delta(U_0)-3\varphi(|U_1|)} C_\varepsilon^{-1/\delta(U_1)} |V|^{1/\delta(U_1)}. \end{aligned}$$

Then (30) yields

$$\mu_\varepsilon(V) \leq 2^{-4j_1\varphi(2^{-j_1})} |V|^{1/\delta(U_1)}.$$

Using the monotonicity of $r \mapsto r^{-\varphi(r)}$, since $|V| \leq 2^{-j_1}$, we see that $|V|^{-\varphi(|V|)} \geq 2^{-j_1\varphi(2^{-j_1})}$ when j_1 is large. This allows us to write

$$\mu_\varepsilon(V) \leq |V|^{1/\delta(U_1)-4\varphi(|V|)}. \tag{31}$$

As in the first step, given $U_1 \in \mathcal{F}_1$, for any pair of distinct elements of $\mathcal{F}_2(U_1)$, namely (V,V'), we have $d(V,V') \leq 2^{-j(U_1)}$.

3.4. **Induction.** Suppose that for $n \geq 2$ we have constructed $\mathcal{F}_0,\dots,\mathcal{F}_n$, a finite sequence of sets of closed dyadic intervals, as well as a measure μ_ε on $\sigma_n = \sigma\Big(I : I \in \bigcup_{1\leq m\leq n}\mathcal{F}_m\Big)$ such that:

(1) For every $1 \leq m \leq n$, each element U of $\mathcal{F}_m$ is included in one element of $\mathcal{F}_{m-1}$, and satisfies $4/\kappa \leq |U|^{-\varphi(|U|)}$.

(2) For every $1 \leq m \leq n$, if $U \in \mathcal{F}_{m-1}$, then there exists a dyadic generation $j(U)$ such that:

   (a) We have

$$2g(U) \leq j(U) \quad \text{and} \quad \#\{V \in \mathcal{F}_m : V \subset U\} = \kappa/4 \cdot 2^{j(U)-g(U)}, \tag{32}$$

   and if two distinct elements V and V' of $\mathcal{F}_m$ belong to U, then $d(V,V') \geq 2^{-j(U)}$.

   (b) for every $V \in \mathcal{F}_m$ such that $V \subset U$, there exist an interval $\widetilde{V} \in \mathcal{Q}(U,j(U),\delta(U)) \setminus \widetilde{\mathcal{Q}}(U,j(U),\delta(U))$ such that $V \subset \widetilde{V} \subset U$, as well as couple $(x(\widetilde{V}),r(\widetilde{V})) \in \mathcal{S}$ satisfying $x(\widetilde{V}) \in \widetilde{V}$ and $2^{-j(U)-1} \leq r(\widetilde{V}) \leq 2^{-j(U)}$. Moreover, every element $x \in V$ is approximated at a rate belonging to $[\delta(U), \delta(U)+\varphi(2^{-j(U)})]$ by $(x(\widetilde{V}),r(\widetilde{V}))$, in the same sense as in (25).

   (c) if $p \geq 1$ satisfies $2^{-g(V)} \leq r_p \leq 2^{-g(U)}$ and $x_p \neq x(\widetilde{V})$, then

$$V \bigcap B\big(x_p, r_p^{\delta(U)+\theta(2^{-j(U)})H_\varepsilon/2}/16\big) = \emptyset.$$

(3) If $1 \leq m \leq n$ and $U \in \mathcal{F}_{m-1}$, then for $V \in \mathcal{F}_m$ such that $V \subset U$:

$$\mu_\varepsilon(V) = \frac{\mu_\varepsilon(U)}{\#\{V' \in \mathcal{F}_m : \ V' \subset U\}}.$$

(4) For all $1 \leq m \leq n$, $U \in \mathcal{F}_{m-1}$ then for $V \in \mathcal{F}_m$ such that $V \subset U$:

$$\mu_\varepsilon(V) \leq |V|^{1/\delta(U)-4\varphi(|V|)}.$$

Parts (1) to (4) of the induction are easily checked for $\mathcal{F}_1$ and $\mathcal{F}_2$.

The technique we use to build the generation $\mathcal{F}_{n+1}$ is the same as for the first iteration. We briefly indicate the steps to follow.

For each $U_n \in \mathcal{F}_n$, there are infinitely many integers $j \geq g(U_n)$ such that

$$\begin{aligned} \#\mathcal{Q}(U_n, j, \delta(U_n)) &\geq \kappa \cdot 2^{j-g(U_n)} \\ \#\widetilde{\mathcal{Q}}(U_n, j, \delta(U_n)) &\leq C \cdot 2^{j-g(U_n)} \Big(2^{-j\varphi(2^{-j})} \\ &\quad + \sum_{g(U_n) \leq k \leq j/\delta(U_n)} 2^{-k(\delta(U_n)-1-\psi(2^{-k}))} \Big). \end{aligned}$$

If the integer $j = j(U_n)$ is large enough, there is a set $\widetilde{\mathcal{F}}_{n+1}(U_n)$ of cardinality $\kappa/4 \cdot 2^{j-g(U_n)}$ included in $\mathcal{Q}(U_n, j, \delta(U_n)) \setminus \widetilde{\mathcal{Q}}(U_n, j, \delta(U_n))$ such that

- the intervals $\widetilde{V}$ of $\widetilde{\mathcal{F}}_{n+1}(U_n)$ are mutually distant from at least $2^{-j(U_n)}$,
- each $\widetilde{V} \in \widetilde{\mathcal{F}}_{n+1}(U_n)$ satisfies $\widetilde{V} \in \mathcal{G}_j$, $\widetilde{V} \subset U_n$, $\mathcal{P}\big(\widetilde{V}, \delta(U_n)\big)$ and

$$\widetilde{V} \bigcap \Big(\bigcup_{k=g(U_n)}^{\gamma(j)} \bigcup_{p \in \mathcal{T}_k} B(y_p, (\rho_p)^{\delta(U_n)}) \Big) = \emptyset.$$

We can associate with each $\widetilde{V} \in \widetilde{\mathcal{F}}_{n+1}(U_n)$ a contracted descendant V, which is a dyadic interval enjoying the properties:

- By property $\mathcal{P}$, there is $x(\widetilde{V}) \in \widetilde{V}$ and a positive real number $r(\widetilde{V})$ satisfying $(x(\widetilde{V}), r(\widetilde{V})) \in \mathcal{S}$ and $2^{-j(U_n)-1} \leq r(\widetilde{V}) \leq 2^{-j(U_n)}$. Moreover, every element $x \in V$ is approximated at a rate belonging to $[\delta(U_n), \delta(U_n) + \varphi(2^{-j(U_n)})]$ by $(x(\widetilde{V}), r(\widetilde{V}))$ (in the sense of (25)),
- if $p \geq 1$ is such that $2^{-g(V)} \leq r_p \leq 2^{-g(U_n)}$ and $x_p \neq x(\widetilde{V})$, then

$$V \bigcap B\big(x_p, (r_p)^{\delta(U_n)+\theta(2^{-j(U_n)})H_\varepsilon/2}/16\big) = \emptyset,$$

- $|V| \geq C_\varepsilon 2^{-j(U_n)\delta(U_n)}$.

Then we set

$$\begin{aligned} \mathcal{F}_{n+1}(U_n) &= \{V : V \text{ is the contracted descendant of some } \widetilde{V} \in \widetilde{\mathcal{F}}_{n+1}(U_n)\}, \\ \text{and} \quad \mathcal{F}_{n+1} &= \{V \in \mathcal{G}_* : \ \exists U_n \in \mathcal{F}_{n+1} \text{ such that } V \in \mathcal{F}_{n+1}(U_n)\}. \end{aligned}$$

The measure μ_ε can be extended into a Borel probability measure on the algebra $\sigma_{n+1} = \sigma(L : L \in \bigcup_{p=0}^{n+1} \mathcal{F}_p)$ by the following formula:

$$\text{for every } U \in \mathcal{F}_{n+1}, \text{ for every } V \in \mathcal{F}_{n+1}(U), \quad \mu_\varepsilon(V) = \frac{\mu_\varepsilon(U)}{\#\mathcal{F}_{n+1}(U)}.$$

In addition, we require that $j_n := \min(j(U_n) : U_n \in \mathcal{F}_n)$ is so large that for all $U \in \mathcal{F}_n$ and $T \in \mathcal{F}_{n-1}$ such that $U \subset T$, we have $j(U) \geq 2g(U)$ and

$$\max\Big(C_\varepsilon^{-1/\delta(U)}, 4/\kappa, 2^{g(U)}, |U|^{1/\delta(T)-3\varphi(|U|)} \Big) \leq 2^{j_n \varphi(2^{-j_n})}.$$

Finally the same lines of computations as in the second step of the construction yield part (4) of the induction, i.e. the scaling behavior of the measure μ_ε on the dyadic intervals of the $(n+1)$th generation of the Cantor set.

Iterating the previous construction, the Kolmogorov theorem yield a measure μ_ε on the algebra $\sigma\big(V:\ V\in\bigcup_{n\geq 1}\mathcal{F}_n\big)$ such that properties (1) to (4) hold for all $n\geq 1$. By construction, μ_ε is carried by the Cantor set

$$\mathscr{K}_\varepsilon=\bigcap_{n\geq 0}\ \bigcup_{V\in\mathcal{F}_n} V.$$

3.5. **Scaling properties of μ_ε.** Let $\delta_\varepsilon=\sup\big\{\delta(U):U\in\bigcup_{n\geq 0}\mathcal{F}_n\big\}$. By (21), we have $\delta_\varepsilon\leq H_\varepsilon:=f(x_\varepsilon)+\varepsilon(2\frac{1}{h^2}+1)$.

We are going to show that there exists $C'>0$ such that

$$\text{(33)}\qquad \text{for every open interval } B\subset[0,1],\quad \mu_\varepsilon(B)\leq C'|B|^{1/\delta_\varepsilon}|B|^{-4\varphi(|B|)}.$$

If (33) holds, then Lemma 3.3, known as the *mass distribution principle* [17], allows to bound from below the lower Hausdorff dimension of μ_ε.

Lemma 3.3. *Let μ be a Borel probability measure on $\mathbb{R}$. Suppose that, for some $\eta>0$, there are $\alpha>0$ and a non-decreasing continuous function ζ such that $\liminf_{x\to 0^+}\frac{\zeta(x)}{x^\alpha}>0$ and for every open interval B with a diameter less than η, $\mu(B)\leq\zeta(|B|)$. Then $\dim_{\mathcal{H}*}(\mu)\geq\alpha$.*

Let B be an open subinterval of $[0,1]$ intersecting $\mathscr{K}_\varepsilon$. Let n_0 be the smallest integer such that B intersects at least two elements of $\mathcal{F}_{n_0}$. By construction, the elements V of $\mathcal{F}_{n_0}$ intersecting B are all contained in the same element U of $\mathcal{F}_{n_0-1}$, and $\mu_\varepsilon(B)\leq\mu_\varepsilon(U)$.

Suppose first that $|B|\geq|U|$. Part (4) of the induction yields

$$\mu_\varepsilon(B)\leq\mu_\varepsilon(U)\leq|U|^{1/\delta_\varepsilon-4\varphi(|U|)}\leq|B|^{1/\delta_\varepsilon-4\varphi(|B|)}$$

when $|B|$ is small. Once again the monotonicity of $r\mapsto r^{-\varphi(r)}$ is used.

Suppose now that $|B|<|U|$. Applying Part (3) of the induction, we find

$$\mu_\varepsilon(B)\leq\mu_\varepsilon(U)\frac{\#\{V\in\mathcal{F}_{n_0}:V\subset U,\ V\cap B\neq\emptyset\}}{\#\{V\in\mathcal{F}_{n_0}:\ V\subset U\}}.$$

Let us use now Part (2) of the induction to bound by above $\#\{V\in\mathcal{F}_{n_0}: V\subset U,\ V\cap B\neq\emptyset\}$. There exists an integer $j(U)$ such that the elements of $\mathcal{F}_{n_0}$ that intersect B are distant from one another by at least $2^{-j(U)}$ and have diameter less than $2^{-j(U)}$. Consequently, by a simple argument, there are at most $2|B|2^{j(U)}$ of them.

In addition, we know by (32) that

$$\#\{V\in\mathcal{F}_{n_0},\ V\subset U\}\geq\kappa/4\cdot 2^{j(U)}2^{-g(U)}\ =\kappa/4\cdot|U|2^{j(U)}.$$

This yields

$$\mu_\varepsilon(B)\quad\leq\quad\mu_\varepsilon(U)\frac{2|B|2^{j(U)}}{\kappa/4|U|2^{j(U)}}\leq 8\kappa^{-1}\cdot\mu_\varepsilon(U)\frac{|B|}{|U|}.$$

Using the scaling behavior of μ_ε on the elements of $\mathcal{F}_{n_0-1}$, we get

$$\begin{aligned}\mu_\varepsilon(B) &\leq 8\kappa^{-1}\cdot|U|^{1/\delta_\varepsilon-4\varphi(|U|)}\frac{|B|}{|U|}\\ &\leq 8\kappa^{-1}\cdot|B|^{1/\delta_\varepsilon-4\varphi(|B|)}\Big(\frac{|B|}{|U|}\Big)^{1-1/\delta_\varepsilon}\frac{|B|^{4\varphi(|B|)}}{|U|^{4\varphi(|U|)}}\\ &= O\big(|B|^{1/\delta_\varepsilon-4\varphi(|B|)}\big),\end{aligned}$$

the last line following from the observation that $\Big(\frac{|B|}{|U|}\Big)^{1-1/\delta_\varepsilon}\frac{|B|^{4\varphi(|B|)}}{|U|^{4\varphi(|U|)}}$ is bounded due to the monotonicity property of $r^{\varphi(r)}$ and to $|B|<|U|$.

By Lemma 3.3, the Hausdorff dimension of μ_ε (and thus the Hausdorff dimension of $\mathscr{K}_\varepsilon$) is larger than $\frac{1}{\delta_\varepsilon}$, which by (21) verifies

$$\begin{aligned}\frac{1}{\delta_\varepsilon} &\geq \frac{1}{H_\varepsilon}\geq\frac{\delta}{f(y_\varepsilon)+\varepsilon(2\frac{1}{h^2}+1)}\geq\frac{1}{f(y_\varepsilon)}\cdot\frac{1}{1+\varepsilon(2\frac{1}{h^2}+1)/f(y_\varepsilon)}\\ &\geq (h-\varepsilon)\cdot\frac{1}{1+\varepsilon(2\frac{1}{h^2}+1)/f(y_\varepsilon)}:=h_\varepsilon. \qquad (34)\end{aligned}$$

It is straightforward that h_ε increases to h when $\varepsilon\to 0$, hence the result.

3.6. **Relation between $\mathscr{K}_\varepsilon$ and $\widetilde{\mathcal{L}}(\Omega,f)$.** Let us prove that $\mathscr{K}_\varepsilon\subset\widetilde{\mathcal{L}}(\Omega,f)$. This justifies all the work we have done!

Let $x\in\mathscr{K}_\varepsilon$ and for $n\geq 1$ denote by $U_n(x)$ the unique element of $\mathcal{F}_n$ that contains x. Using parts (2) and (3) of the induction, we have $\delta_x=\limsup_{n\to\infty}\delta(U_n(x))$.

Recall that the function f is continuous at x. Using formula (19), one observes that $f(y_{U_n(x)})$ converges to $f(x)$ (since $y_{U_n(x)}$ is any point of $U_n(x)$). This implies that $\delta(U_n(x))$ converges to $f(x)$ when n tends to infinity.

Finally, $\delta_x=\lim_{n\to\infty}\delta(U_n(x))=f(x)$.

3.7. **The general case.** We have treated the case where f is continuous and $\inf\{f(x):x\in\Omega\}>1$.

We explain here how to adapt the result to the case where f is continuous except at points of a subset E of Ω of Hausdorff dimension strictly less than $\inf\{f(x):x\in\Omega\setminus E\}$. We suppose that

$$h:=(\inf\{f(x):x\in\Omega\setminus E\})^{-1}>\dim_{\mathcal{H}}E. \qquad (35)$$

Also, it is worth recalling that property (4) combined with the weak redundancy assumption on the irreducible system deduced from $\mathcal{S}$ implies that for each subinterval B of $[0,1]$, we have $\ell(\widetilde{\mathcal{L}}_1(\mathcal{S})\cap B)=|B|$ (see [4] for details).

- Suppose that there exists a non trivial subinterval B of Ω such that the restriction of f to $B\setminus E$ is equal to 1. Then, we have $h=1$, and due to the information recalled above, we have $\ell(\widetilde{\mathcal{L}}(B\setminus E,f))=\ell(\widetilde{\mathcal{L}}_1(\mathcal{S})\cap B)=|B|$. This yields the desired Hausdorff dimension.

• Now we assume that $h<1$. We modify the selection of the intervals Ω_ε at the beginning of the proof (Section 3.1) as follows: Fix $\varepsilon \in (0,h)$. By (35), and since $\dim_{\mathcal{H}} E < 1 = \dim_{\mathcal{H}} \Omega$, there exists $y_\varepsilon \in \overset{\circ}{\Omega} \setminus E$ such that $h - \varepsilon/2 \leq \dfrac{1}{f(y_\varepsilon)} \leq h$. Hence, using the continuity of f at y_ε, for every y in a neighborhood $\Omega_\varepsilon \subset \overset{\circ}{\Omega}$ small enough around y_ε, we have $h-\varepsilon \leq \dfrac{1}{f(y)} \leq h$. Hence, when ε is small, (18) holds, and the proof is not modified afterwards.

• The way we conclude to find the lower bound for $\dim_{\mathcal{H}} \widetilde{\mathcal{L}}(\Omega \setminus E, f)$ differs slightly from the continuous case. Assume that we have performed the construction of μ_ε and $\mathscr{K}_\varepsilon$. Since $\dim_{\mathcal{H}} E < h$, when ε is small enough we have $\mu_\varepsilon(E)=0$. Recalling that the support of μ_ε is $\mathscr{K}_\varepsilon$, we deduce that

$$\dim_{\mathcal{H}} \widetilde{\mathcal{L}}(\Omega \setminus E, f) \geq \dim_{\mathcal{H}} \mathscr{K}_\varepsilon \setminus E \geq \dim_{\mathcal{H}}(\mu_\varepsilon) \geq h_\varepsilon.$$

Letting ε tend to 0 yields

$$\dim_{\mathcal{H}} \widetilde{\mathcal{L}}(\Omega \setminus E, f) \geq h.$$

• The last possible situation is $h := \inf\{f(x) : x \in \Omega \setminus E\} = 1$, and there is no non-trivial subinterval B of Ω such that f is equal to 1 over $B \setminus E$. For every $\varepsilon>0$ small enough, it is possible to find a dyadic subinterval U_ε of Ω such that the restriction of f to $U_\varepsilon \setminus E$ has an infimum which belongs to the open interval $(1, 1+\varepsilon)$. The modification of the proof explained above can be applied to the set $\widetilde{\mathcal{L}}(U_\varepsilon \cap (\Omega \setminus E), f)$, and we find that $\dim_{\mathcal{H}} \widetilde{\mathcal{L}}(U_\varepsilon \cap (\Omega \setminus E), f) \geq 1/(1+\varepsilon)$. Letting ε tend to zero yields the result.

3.8. **Density of $\widetilde{\mathcal{L}}(\Omega \setminus E, f)$ in Ω when $\dim_{\mathcal{H}} E < \frac{1}{\sup\{f(x):x\in\Omega\setminus E\}}$.** Using what precedes, we are able to construct a Cantor set $\mathcal{K}_\varepsilon$ in order to approximate the Hausdorff dimension of $\widetilde{\mathcal{L}}(\Omega \setminus E, f)$. But our construction may be achieved in a neighborhood U_y of any point $y \in \Omega \setminus E$ such that $\dim_{\mathcal{H}} E \cap U_y < \frac{1}{\inf\{f(x):x\in U_y\setminus E\}}$. Consequently, if $\dim_{\mathcal{H}} E < \frac{1}{\sup\{f(x):x\in\Omega\setminus E\}}$ then we get the conclusion, since $\Omega \setminus E$ is dense in Ω.

4. Upper bounds for the dimensions

We suppose that the assumptions of Theorem 1.7 are fulfilled. As in the previous section, we set $\delta = \inf\{f(x) : x \in \Omega \setminus E\}$ and $h = 1/\delta$.

By (9), we know that $\delta_x \geq 1$ for all $x \in \Omega$. The set $\widetilde{\mathcal{L}}(\Omega \setminus E, f)$ contains only elements $x \in [0,1]$ satisfying $f(x) \geq \delta$, which implies that $\delta(x) \geq \delta$. Hence, for every $\varepsilon > 0$, $\widetilde{\mathcal{L}}(\Omega \setminus E, f) \subset \mathcal{L}_{\delta-\varepsilon}(\mathcal{S})$, where (recall (2)):

$$\mathcal{L}_{\delta'}(\mathcal{S}) = \bigcap_{N\geq 1} \bigcup_{n\geq N} B(x_n, r_n^{\delta'}) = \bigcap_{N\geq 1} \bigcup_{n\geq N} B(y_n, \rho_n^{\delta'}).$$

It is known (see [4]) that if the system $\mathcal{S}$ is weakly redundant, then $\dim_{\mathcal{H}} \mathcal{L}_\delta(\mathcal{S}) \leq 1/\delta$ for all $\delta \geq 1$. Let us prove it briefly for completeness.

Let $s > 1/\delta$. For any integer $N \geq 1$, a covering of the limsup set $\mathcal{L}_\delta(\mathcal{S})$ is provided by the union of sets $\bigcup_{n\geq N} B(y_n, \rho_n^\delta)$. Let $\eta > 0$, and choose N large enough so that $2\rho_n \leq \eta$ for $n \geq N$. Recalling the definition of the s-Hausdorff measure, we see that (the sets $\mathcal{T}_j$ appear in Definition 2.3)

$$\mathcal{H}_\eta^s(\mathcal{L}_\delta(\mathcal{S})) \leq \sum_{n\geq N} |B(y_n, \rho_n^\delta)|^s \leq \sum_{j=J}^{+\infty} \sum_{p\in\mathcal{T}_j} |B(y_p, \rho_p^\delta)|^s \leq \sum_{j=J}^{+\infty} \sum_{p\in\mathcal{T}_j} 2^{-js\delta},$$

where J is the unique integer such that $y_N \in \mathcal{T}_J$. Using the weak redundancy, we see that

$$\mathcal{H}_\eta^{\zeta_s}(\mathcal{L}_\delta(\mathcal{S})) \quad \leq \quad \sum_{j=J}^{+\infty} N_j \cdot 2^{j-js\delta} \leq \sum_{j=J}^{+\infty} N_j \cdot 2^{j(1-s\delta)}.$$

This series converges, since $\log(N_j) = o(j)$ and $1 - s\delta < 0$ by construction. Consequently, the s-Hausdorff measure of $\mathcal{L}_\delta$ is finite for any $s > 1/\delta$. This demonstrates that $\dim_{\mathcal{H}} \mathcal{L}_\delta(\mathcal{S}) \leq 1/\delta$.

The above argument applies to $\mathcal{L}_{\delta-\varepsilon}(\mathcal{S})$ when $\delta - \varepsilon > 1$. Since $\widetilde{\mathcal{L}}(\Omega \setminus E, f) \subset \bigcap_{\varepsilon>0} \mathcal{L}_{\delta-\varepsilon}(\mathcal{S})$, we have

$$\dim_{\mathcal{H}}\widetilde{\mathcal{L}}(\Omega \setminus E, f) \leq \inf_{\varepsilon>0} \dim_{\mathcal{H}} \mathcal{L}_{\delta-\varepsilon}(\mathcal{S}) \leq \inf_{\varepsilon>0} 1/(\delta-\varepsilon) = 1/\delta = h.$$

5. Examples of suitable systems $\big((x_n, r_n)\big)_{n\geq 1}$

5.1. **Approximation by b-adic numbers.** We prove that the dyadic system satisfies $\mathcal{P}$. The b-adic system (whose definition is clear) is similar.

Define the system $\mathcal{D} = \big((k2^{-j}, 2^{-j})\big)_{j\geq 1, k\in\{1,\dots,2^{-j}-1\}}$, and consider the approximation exponent of any $x \in [0,1]$ by $\mathcal{D}$

$$\delta_x = \sup\{\delta \geq 1 : |x - k2^{-j}| \leq 2^{-j\delta} \text{ for an infinite number of } (j,k)\}.$$

We rather consider the system $\mathcal{D}' = \big((k2^{-j}, 2^{-j}/32)\big)_{j\geq 1, k\in\{1,\dots,2^{-j}-1\}}$ and the associated approximation exponent

$$\delta'_x = \sup\big\{\delta \geq 1 : |x - k2^{-j}| \leq (2^{-j}/32)^\delta \text{ for an infinite number of } (j,k)\big\}.$$

Of course, $\delta_x = \delta'_x$ for every $x \in [0,1]$, but the constant 32 is necessary for our property $\mathcal{P}$ to hold.

Proposition 5.1. *The system $\mathcal{D}'$ satisfies the property $\mathcal{P}$.*

Proof. The irreducible system of $\mathcal{D}'$ consists in the couples $(k2^{-j}, 2^{-j}/32)$ for which k is odd. Therefore, it is obvious that the weak redundancy is satisfied, the corresponding sequence $(N_j)_{j\geq 1}$ being constant equal to 1, so that $\gamma(j) = j$ for every $j \geq 1$.

To check $\mathcal{P}$, let $\delta > 1$, and consider any $\varphi \in \Phi$. Let $k2^{-j}$ be a dyadic element of $[0,1]$ such that k is odd, and set $V := [k2^{-j}, (k+1)2^{-j})$. Given a dyadic generation j, the number of such dyadic irreducible intervals is 2^{j-1}. The property $\mathcal{P}(V, \delta)$ holds without any further condition. Indeed, we only have to check that for every $j' \in \{\gamma(j), ..., (j+1)\delta+4\}$, for every odd integer k', $k'2^{-j'} \notin B(k2^{-j}, (2^{-j}/32)^\delta)$. This is obvious, since by the structure of the dyadic tree we get when $j \leq j' \leq (j+1)\delta + 4$

$$|k2^{-j} - k'2^{-j'}| \ \geq \ 2^{-j'} \geq 2^{-(j+1)\delta-4} \geq 2^{-j\delta}/(16 \cdot 2^\delta) \geq (2^{-j}/32)^\delta,$$

and when $\gamma(j) \leq j' < j$

$$|k2^{-j} - k'2^{-j'}| \geq 2^{-j} \geq (2^{-j}/32)^\delta. \qquad \square$$

Thus the system $\mathcal{D}$ satisfies $\mathcal{P}$ with a function κ constant equal to 1/2 (it holds for all the "irreducible" sub-intervals of $[0,1]$).

5.2. **Diophantine approximation by rational numbers.** Consider

$$\mathcal{R} = \big((p/q, 1/q^2)\big)_{q\geq 1,\, 1\leq p\leq q-1}.$$

It follows from Dirichlet's argument that $\mathcal{L}_1(\mathcal{R}) = [0,1]$. The irreducible sub-system of $\mathcal{R}$ consists in the elements of $\mathcal{R}$ such that $p \wedge q = 1$.

Proposition 5.2. *The system $\mathcal{R}$ satisfies the property $\mathcal{P}$.*

Proof. Let $j \geq 1$ be an integer, and let $(p/q, 1/q^2) \in \mathcal{R}$ be such that $q^2 \in (2^j, 2^{j+1}]$. We shall prove that $B(p/q, 1/q^2)$ may contain only a bounded number of rational numbers p'/q' satisfying $(p'/q', 1/(q')^2) \in \mathcal{R}$ and $(q')^2 \in (2^j, 2^{j+1}]$. This implies the weak redundancy.

If $p/q \neq p'/q'$, then one has necessarily that $|p/q - p'/q'| = |pq' - p'q|/(qq') \geq 1/(qq') \geq 2^{-j-1}$, since q and q' belong to $[2^{j/2}, 2^{(j+1)/2})$. Since the diameter of $B(p/q, 1/q^2)$ is at most 2^{-j+1}, there are at most 4 distinct irreducible rational numbers p'/q' belonging to $B(p/q, 1/q^2)$. Hence $\mathcal{R}$ is weakly redundant, with a sequence $(N_j)_{j\geq 1}$ constant equal to 4.

In order to prove $\mathcal{P}$, we consider $V := [K \cdot 2^{-J}, (K+1) \cdot 2^{-J})$ a dyadic interval in $[0,1]$ of generation J, a real number $\delta > 1$ and any function $\varphi \in \Phi$. We demonstrate that $\mathcal{P}(V, \delta)$ holds without any restriction on V, δ and φ. Obviously V contains a rational number p/q satisfying $q^2 \in (2^J, 2^{J+1}]$ (p/q is not necessarily irreducible). Assume that a rational number $p'/q' \neq p/q$ belongs to $B(p/q, 1/q^{2\delta})$ with $\log_2((q')^2) \in [\gamma(J), \cdots, \delta(J+1)+4]$. This implies that $q' \leq 2^{(\delta(J+1)+4)/2} \leq q^{\delta/2} 2^{\delta/2+2}$. Combining the information, we have

$$1/q^{2\delta} \geq |p/q - p'/q'| \geq 1/(qq') \geq 2^{-\delta/2-2}/q^{1+\delta/2}.$$

This last inequalities can not hold as soon as $\delta > 1$ (provided that q is large enough). Consequently, $\mathcal{P}(V, \delta)$ holds, and $\mathcal{R}$ satisfies $\mathcal{P}$ with a function κ constant equal to 1. $\square$

5.3. **Inhomogeneous Diophantine approximation.** Let α be an irrational number in $[0,1]$. Consider the system

$$\mathcal{I} = \Big(\big(\{n\alpha\}, \frac{1}{n}\big)\Big)_{n\geq 1},$$

where $\{x\}$ stands for the fractional part of the real number x.

Proposition 5.3. *The system $\mathcal{I}$ satisfies the property $\mathcal{P}$.*

Proof. It is proved in [4] (Proposition 6.1) that $\mathcal{I}$ is weakly redundant if and only if the approximation exponent of α by the rational system $\mathcal{R}$ equals 2.

We prove that $\mathcal{I}$ satisfies $\mathcal{P}$, when the approximation exponent of α by the rational system $\mathcal{R}$ is 2. When this holds, for every $\varepsilon > 0$, there is an integer q_ε such that

$$\text{for every } q \geq q_\varepsilon, \text{ for every integer } p, \ |\alpha - p/q| \geq 1/q^{1+\varepsilon}. \tag{36}$$

We focus on $\mathcal{P}$. For this, let us recall the *three distance theorem* [36, 35, 14]:

Theorem 5.4. *The real numbers $\{\alpha\}$, $\{2\alpha\}$, $\{3\alpha\}$, ..., $\{N\alpha\}$ divide the interval $[0,1]$ into $N+1$ intervals whose lengths take at most three values $d_1(N)$, $d_2(N)$ and $d_3(N)$, satisfying*

$$d_1(N) < d_2(N) < d_3(N) \leq \frac{3}{N+1}.$$

Let $J \geq 1$. As for the rational system, in order to prove $\mathcal{P}$, we consider $V := [K \cdot 2^{-J}, (K+1) \cdot 2^{-J})$ a dyadic interval in $[0,1]$ of generation J, a real number $\delta > 1$ and any function $\varphi \in \Phi$. We demonstrate that $\mathcal{P}(V,\delta)$ holds without any restriction on V, δ and φ for a sufficiently large number of dyadic intervals V.

Apply the three distance theorem to $\{\alpha\}$, $\{2\alpha\}$, $\{3\alpha\}$, ..., $\{2^J\alpha\}$. The 2^J+1 corresponding intervals of $[0,1]$ have length less than $3/(2^J+1)$. By a translation argument, the points $\{(2^J+1)\alpha\}$, $\{(2^J+2)\alpha\}$, $\{(2^J+3)\alpha\}$, ..., $\{2^{J+1}\alpha\}$ divide the interval $[0,1]$ into 2^J+1 intervals whose lengths are also less than $3/(2^J+1)$. This means that among the dyadic intervals of generation J, there are no three consecutive dyadic intervals U which do not contain one of the points $\{n\alpha\}$, for n ranging over $\{2^J+1, 2^J+2, ..., 2^{J+1}\}$.

Let us consider one such interval $V := [K\cdot 2^{-J}, (K+1)\cdot 2^{-J})$, which contains $\{n\alpha\}$ for some n belonging to $\{2^J+1, 2^J+2, ..., 2^{J+1}\}$. Assume that another point $\{n'\alpha\}$ belongs to $B(\{n\alpha\}, 1/n^\delta)$ with $\log_2 n' \in [\gamma(J), \cdots, [\delta(J+1)]+4]$. This means that $|\{n'\alpha\} - \{n\alpha\}| \leq \frac{1}{n^\delta}$. By definition there are integers p and p' satisfying $n\alpha = p + \{n'\alpha\}$ and $n'\alpha = p' + \{n\alpha\}$, hence

$$|(n'\alpha + p') - (n\alpha - p)| = |(n-n')\alpha - (p'-p)| \leq \frac{1}{n^\delta},$$

or equivalently

$$\left|\alpha - \frac{p'-p}{n'-n}\right| \leq \frac{1}{|n'-n| \cdot n^\delta} \leq \frac{C}{|n'-n|^{\delta+1}},$$

the last inequality following from the fact that $|n'-n| \leq 2\delta \cdot n$. This contradicts (36). Consequently, $\mathcal{P}(V,\delta)$ holds.

Finally, $\mathcal{I}$ satisfies $\mathcal{P}$ with a function κ constant equal to 1/3. □

5.4. **Poisson point process.** Let $\mathcal{S}$ be a Poisson point process with intensity

$$\Lambda = \mathbf{1}_{[0,1]\times(0,1)}(x,y)\cdot \ell(dx)\otimes \frac{\ell(dy)}{y^2}, \tag{37}$$

where ℓ stands for the Lebesgue measure on $(0,1)$. We rewrite $\mathcal{S}$ as $\mathcal{S} = \big((x_n,r_n)\big)_{n\geq 1}$, where $(r_n)_{n\geq 1}$ is a positive decreasing sequence converging to zero when n tends to infinity.

With probability one, such a system is weakly redundant (see Proposition 6.2 in [4]) and satisfies $\mathcal{P}$ (see Section 6 in [2]). In reality, the function κ in $\mathcal{P}(V,\delta)$ has been introduced to deal with the Poisson system, which is very important when studying the local regularity analysis of some Markov processes.

References

[1] V.I. Arnold, Small denominators and problems of stability in classical and celestial mechanics, *Pspekhi Mat. Nauk* 18 (1963) 91–192.

[2] J. Barral, S. Jaffard, N. Fournier, S. Seuret, Markov processes with random singularity spectra. ArXiv math.PR/0907.0104, 2009.

[3] J. Barral, S. Seuret, Combining multifractal additive and multiplicative chaos, *Commun. Math. Phys.* 257 (2005) 473–497.

[4] J. Barral, S. Seuret, Heterogeneous ubiquitous systems in $\mathbb{R}^d$ and Hausdorff dimension, *Bull. Braz. Math. Soc.* 38 (2007) 467–515.

[5] J. Barral, S. Seuret, The singularity spectrum of Lévy processes in multifractal time, *Adv. Math.* 214 (2007) 437–468.

[6] V. Beresnevich, D. Dickinson and S. L. Velani, Sets of exact logarithmic order in the theory of Diophantine approxiamtion, Math. Ann. 321 (2001) 253–273.

[7] V. Beresnevich, H. Dickinson, S. Velani, Measure theoretic laws for limsup sets, *Memoirs of the AMS.* 179 (2006) 1–91.

[8] V. Beresnevich, M.M. Dodson, S. Kristensen, J. Levesley, An inhomogeneous wave equation and non-linear Diophantine approximation, *Adv. Math.* 217 (2008) 740–760.

[9] V. Beresnevich, S. Velani, A Mass Transference Principle and the Duffin-Schaeffer conjecture for Hausdorff measures, *Ann. Math.* 164 (2006) 971–992.

[10] A.S. Besicovitch, Sets of fractional dimension (IV): on rational approximation to real numbers, *J. London Math. Soc.*, 9 (1934), 126–131.

[11] J. D. Bovey, M. M. Dodson, The Hausdorff dimension of systems of linear forms, *Acta Arith.* 45 (1986), 337–358.

[12] Y. Bugeaud, Approximation by algebraic integers and Hausdorff dimension, *J. London Math. Soc.* 65 (2002) 547-559.

[13] Y. Bugeaud, Sets of exact approximation order by rational numbers *Math. Ann.* 327 (2003) 171-190.

[14] Y. Bugeaud, A note on inhomogeneous Diophantine approximation, *Glasgow Math. J.* 45 (2003) 105-110.

[15] M.M. Dodson, M.V, Melián, D. Pestana, S.L. Vélani, Patterson measure and Ubiquity *Ann. Acad. Sci. Fenn. Ser. A I Math.* 20 (1995) 37–60.

[16] H.G. Eggleston, Sets of fractional dimensions which occur in some problems of number theory, *Proc. London Math. Soc.* 54(1952), 42–93.

[17] K. J. Falconer, Techniques in Fractal Geometry (Wiley, New York, 1997).

[18] K. J. Falconer, Sets with large intersection properties, *J. London Math. Soc.* 49 (1994) 267–280.

[19] K. J. Falconer, Representation of families of sets by measures, dimension spectra and Diophantine approximation, *Math. Proc. Cambridge Philos. Soc.* 128 (2000) 111–121.
[20] G.H. Hardy, E.M. Wright, An Introduction to the Theory of Numbers (Fifth edition, Oxford Science publications, 1979).
[21] R. Hill, S.L. Velani, The Jarnik-Besicovitch theorem for geometrically finite Kleinian groups, *Proc. Lond. Math. Soc.* 77 (1998) 524–550.
[22] R. Hill, S.L. Velani, Ergodic theory of shrinking targets, *Invent. Math.* 119 (1995) 175-198.
[23] S. Jaffard, The spectrum of singularities of Riemann's function, *Rev. Mat. Iberoamericana* 12 (1996) 441–460.
[24] S. Jaffard, Old friends revisited: The multifractal nature of some classical functions, *J. Fourier Anal. Appl.* 3 (1997) 1–22.
[25] S. Jaffard, On lacunary wavelet series, *Ann. Appl. Probab.* 10 (2000) 313–329.
[26] S. Jaffard, The multifractal nature of Lévy processes, *Probab. Theory Relat. Fields* 114 (1999) 207–227.
[27] V. Jarnik, Diophantischen Approximationen und Hausdorffsches Mass, *Mat. Sbornik* 36 (1929) 371–381.
[28] V. Jarnik, Uber die simultanen diophantischen Approximationen, *Math. Z.* 33 (1931) 503–543.
[29] A.Y. Khintchine, Einige Sätze über Kettenbrüche, mit Anwendungen auf die Theorie der diohantischen Approximationen, *Math. Ann.* 92 (1924) 115–125.
[30] A.Y. Khintchine, Zur metrischen Theorie der diophantischen Approximationen, *Math. Z.* 24 (1926) 706–714.
[31] P. Mattila, Geometry of Sets and Measures in Euclidian Spaces. (Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1995).
[32] G. Petronilho, Global solvavility and simultaneous approximable vectors, *J. Differential Equations* 184 (2002), 48–61.
[33] G. Petronilho, Global s-solvability, global s-hypoellipticity and Diophantine approximation, *Indag. Math. (N.S.)* 16 (2005) 67–90.
[34] B.I. Ptashnik, Improper Boundary Problems for Partial Differential Equations. (Nauka Dumka, 1984).
[35] N.B. Slater, Gaps and steps for the sequence $n\theta \mod 1$, *Proc. Cambridge Philos. Soc.* 63 (1967) 1115–1123.
[36] V.T. Sós, On the theory of Diophantine approximation I, *Acta Math. Hung.* 8 (1957) 461–472.
[37] B.O. Stratmann, M. Urbanski, Jarnik and Julia: a Diophantine analysis for parabolic rational maps, *Math. Scan.* 91 (2002) 27–54.

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