A LOCALIZED JARNIK-BESICOVITCH THEOREM

JULIEN BARRAL AND STÉPHANE SEURET

Abstract. Fundamental questions in Diophantine approximation are related to the Hausdorff dimension of sets of the form $\{x \in \mathbb{R} : \delta_x = \delta\}$, where $\delta \geq 1$ and δ_x is the Diophantine approximation rate of an irrational number x. We go beyond the classical results by computing the Hausdorff dimension of the sets $\{x \in \mathbb{R} : \delta_x = f(x)\}$, where f is a continuous function. Our theorem applies to the study of the approximation rates by various approximation families. It also applies to functions f which are continuous outside a set of prescribed Hausdorff dimension.

1. Introduction

Let $(x_n)_{n\geq 1}$ be a sequence in a σ-compact metric space (E, d), and $(r_n)_{n\geq 1}$ be a non-increasing sequence of real numbers converging to 0 when n tends to infinity. Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a positive non-decreasing continuous mapping with $\varphi(0) = 0$. The set

$$\mathcal{L}(\varphi) = \{x \in E : \ d(x, x_n) \leq \varphi(r_n) \text{ for infinitely many integers } n\} \tag{1}$$

contains the elements of E that are infinitely often well-approximated, at rate φ, by the points x_n relatively to the radii r_n. This set can be rewritten as a limsup set:

$$\mathcal{L}(\varphi) = \limsup_{n\to\infty} B\big(x_n, \varphi(r_n)\big).$$

($B(x, r)$ stands for the ball of centre x and radius r.) The values of the Hausdorff dimensions (or the Hausdorff measure associated with convenient gauge functions) of sets of the form $\mathcal{L}(\varphi)$ provide us with a fine description of the geometrical distribution in E of the sequence $(x_n)_{n\geq 1}$.

Such limsup sets arise naturally in Diophantine approximation theory in $\mathbb{R}^d$ (see [30, 28, 31, 10, 20, 12, 13, 3, 4, 39] among many references), and more generally in Diophantine approximation problems in limit sets of groups or in Julia sets of rational maps [15, 38, 22, 23]. They also appear in mathematical physics and dynamical systems when studying resonance problems [1, 33, 34, 35, 8], and when measuring the distribution of Hölder singularities of measures and functions [24, 27, 25, 19, 5, 6].

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We denominate the sequence of couples $\mathcal{S} = \big((x_n, r_n)\big)_{n\geq 1}$ as an *approximation system* (or simply a *system*) in E. Standard Diophantine approximation deals with the approximation of real numbers by the system $\big((p/q, 1/q^2)\big)_{q\geq 1,\, p\in\mathbb{Z}}$.

The mappings φ of the form $\varphi_\delta : r \mapsto r^\delta$ (for $\delta > 0$) are particularly relevant. Hence, denoting $\mathcal{L}(\varphi_\delta)$ simply by $\mathcal{L}_\delta$, we consider

$$\mathcal{L}_\delta = \Big\{ x \in E : d(x, x_n) \leq r_n^\delta \text{ for infinitely many integers } n \Big\}. \tag{2}$$

The sets $(\mathcal{L}_\delta)_{\delta>0}$ form a non-increasing family of sets. This property allows us to classify the elements x of E according to their *approximation rate* by the system $\mathcal{S} = \big((x_n, r_n)\big)_{n\geq 1}$. This approximation rate is defined for $x \in E$ by (we use the convention that $\sup \emptyset = 0$)

$$\delta_x = \sup\{\delta : x \in \mathcal{L}_\delta\},$$

and for $\delta \geq 0$, one is naturally interested in the set $\widetilde{\mathcal{L}}_\delta$ of points which have approximation rate δ:

$$\widetilde{\mathcal{L}}_\delta = \{x \in E : \ \delta_x = \delta\}. \tag{3}$$

We emphasize the following embedment properties between the sets $\mathcal{L}_\delta$ and $\widetilde{\mathcal{L}}_\delta$: For $\delta > 0$,

$$\mathcal{L}_\delta = \bigcup_{\delta'\geq\delta} \widetilde{\mathcal{L}}_{\delta'} \quad \text{and} \quad \widetilde{\mathcal{L}}_\delta = \bigcap_{\delta'\geq\delta} \mathcal{L}_{\delta'} \setminus \bigcup_{\delta'<\delta} \mathcal{L}_{\delta'}. \tag{4}$$

The dimension problems related with the sets $\mathcal{L}_\delta$ are also relevant for the sets $\widetilde{\mathcal{L}}_\delta$. Hence, given $\delta > 0$, it is natural to question the non-emptiness of $\widetilde{\mathcal{L}}_\delta$, and the value of the Hausdorff dimension of $\widetilde{\mathcal{L}}_\delta$, and the existence of gauge functions ζ for which the corresponding Hausdorff measure $\mathcal{H}^\zeta(\widetilde{\mathcal{L}}_\delta)$ is null, positive and finite, or infinite ($\mathcal{H}^\xi$ stands for the generalized Hausdorff measure associated with the gauge function ξ, see Section 2.1). The first investigations on this subject have led to the celebrated Jarnik-Besicovitch theorem: if the system $\mathcal{S}$ is the rational system $\big((p/q, 1/q^2)\big)_{q\geq 1,\, p\in\mathbb{Z}}$, then for every $\delta \geq 1$, $\dim_{\mathcal{H}} \mathcal{L}_\delta = 1/\delta$ ($\dim_{\mathcal{H}}$ stands for the Hausdorff dimension).

Note that in the case of the approximation by rational numbers, $\mathcal{L}_1 = \mathbb{R}$, since by a famous Dirichlet's result, for every $x \in \mathbb{R}$, the event $x \in B(p/q, 1/q^2)$ occurs for infinitely many integers $q \geq 1$. In fact, in many situations, the system $\mathcal{S}$ is chosen so that the set

$$\mathcal{L}_1 = \limsup_{n\to\infty} B(x_n, r_n) \text{ is of full } m\text{-measure in } E, \tag{5}$$

where m is a probability measure on E enjoying nice scaling properties:

- In [26, 15], m is *uniformly distributed*: There exist $r_0 > 0$ and $C > 1$ such that $C^{-1} r^{\dim_{\mathcal{H}} E} \leq m(B(x,r)) \leq C r^{\dim_{\mathcal{H}} E}$ for all $x \in E$ and $r \in (0, r_0]$. Observe that if such a measure m exists on E, then

$\dim_{\mathcal{H}}(m) = \dim_{\mathcal{H}}(E)$ ($\dim_{\mathcal{H}}(m)$ is the Hausdorff dimension of the measure m).

- In [3], m is supposed to possess deterministic or statistical self-similarity properties, which imply the weaker property: There exists $\delta \in (0, \dim_{\mathcal{H}} E]$ such that $\lim_{r\to 0^+} \frac{\log(m(B(x,r))}{\log(r)} = \dim_{\mathcal{H}}(m)$ for m-almost every x.

In these contexts, it can be proved that for every $\delta \geq 1$, $\dim_{\mathcal{H}} \mathcal{L}_\delta \geq \dim_{\mathcal{H}}(m)/\delta$. Moreover, there exists a gauge function $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\lim_{r\to 0^+} \frac{\log \xi(r)}{\log r} = \dim_{\mathcal{H}}(m)/\delta$ and $\mathcal{H}^\xi(\mathcal{L}_\delta) > 0$. Such theorems are referred to as *ubiquity* results, and the literature on ubiquity properties is numerous.

From now on, we focus on $\mathbb{R}^d$, with $d \geq 1$, and more precisely, for obvious periodicity reason, on $E = [0,1]^d$.

As mentioned previously, the authors of [26, 15] obtained the following ubiquity theorem which treats the case where the measure m in (5) equals ℓ, the Lebesgue measure in $\mathbb{R}^d$. Theorem 1.1 establishes an extension of the famous Jarnik-Besicovitch theorem for Diophantine approximation by rational numbers.

Theorem 1.1. *Let* $\mathcal{S} = \Big((x_n, r_n)\Big)_{n\geq 1}$ *be a system in* $[0,1]^d$. *If*

$$\ell\Big(\mathcal{L}_1\Big) = 1, \tag{6}$$

then for every $\delta \geq 1$, $\dim_{\mathcal{H}} \mathcal{L}_\delta \geq d/\delta$.

In addition, there exists a gauge function $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ *satisfying* $\lim_{r\to 0^+} \frac{\log \xi(r)}{\log r} = d/\delta$ *and* $\mathcal{H}^\xi(\mathcal{L}_\delta) > 0$.

The second part of Theorem 1.1 is crucial, since it deals with Hausdorff measures (and not only with the Hausdorff dimension). It makes it possible to replace the set $\mathcal{L}_\delta$ by $\widetilde{\mathcal{L}}_\delta$ in the statement of Theorem 1.1, provided that the balls $B(x_n, r_n)$ with comparable diameters do not overlap excessively. This occurs when there exists an integer $N > 0$ such for all $j \geq 0$, each element $x \in [0,1]^d$ belongs to at most N balls $B(x_n, r_n)$ such that $2^{-j-1} \leq r_n \leq 2^{-j}$ (heuristically, the elements of $[0,1]^d$ are covered "economically" by the balls $B(x_n, r_n)$). Such a property is a specific case of the *weak redundancy* property $\mathcal{C}_1$, which will be defined in Section 2.3. In this case, we thus have for all $\delta \geq 1$

$$\dim_{\mathcal{H}} \mathcal{L}_\delta = \dim_{\mathcal{H}} \widetilde{\mathcal{L}}_\delta = \frac{d}{\delta}. \tag{7}$$

This two-sided equality contains two results: the non-emptiness of $\widetilde{\mathcal{L}}_\delta$ and the value of its Hausdorff dimension. It (7) holds in $\mathbb{R}$ when considering the "rational" system $\mathcal{R} = \big((p/q, 1/q^2)\big)_{q\geq 1,\, 0<p<q}$ or other systems of points (for instance obtained as Poisson point processes in the upper-half-plane, see Section 5 for details and further examples).

Finally, observe that for a given system $\mathcal{S} = \big((x_n, r_n)\big)_{n\geq 1}$, the covering property

$$\limsup_{n\to\infty} B(x_n, r_n) = (0,1)^d \tag{8}$$

implies (6), and the corresponding approximation rates satisfy

$$\delta_x \geq 1 \quad \text{for all } x \in (0,1)^d.$$

Therefore, the associated sets $\widetilde{\mathcal{L}}_\delta$ provide us with a classification of all the elements of $[0,1]^d$ with respect to their approximation rates (those associated with $\mathcal{S}$).

In this article we replace the (constant) approximation rate δ in (3) by $f(x)$, where f is a *continuous function*. We are thus looking for elements $x \in [0,1]^d$ whose *approximation rate by some system $\mathcal{S}$ depend on x via the function f*. Hence, Jarnik-Besicovitch's Theorem and the result on the Hausdorff dimension of $\widetilde{\mathcal{L}}_\delta$ in Theorem 1.1 will be viewed as Theorem 1.2 in the special case where f is a constant function.

Let us state our main theorem. Conditions $\mathcal{C}_1$ and $\mathcal{C}_2$ will be explained later.

Theorem 1.2. *Consider the system $\mathcal{S} = \big((x_n, r_n)\big)_{n\geq 1}$, where $(x_n)_{n\geq 1}$ is a sequence of elements of $(0,1)^d$ and $(r_n)_{n\geq 1}$ is a non-increasing sequence of real numbers converging to 0 when n tends to infinity.*

Assume that (8), $\mathcal{C}_1$ and $\mathcal{C}_2$ hold.

Let Ω be a non-empty compact subset of $(0,1)^d$, such that $\overline{\overset{\circ}{\Omega}} = \Omega$.

Let $f : (0,1)^d \to [1,+\infty)$ be a continuous function.

Consider the subsets of $[0,1]^d$ defined by

$$\mathcal{L}(\Omega, f) \;=\; \{x \in \Omega : \ \delta_x \geq f(x)\} \tag{9}$$

$$\widetilde{\mathcal{L}}(\Omega, f) \;=\; \{x \in \Omega : \ \delta_x = f(x)\} \tag{10}$$

The sets $\mathcal{L}(\Omega, f)$ and $\widetilde{\mathcal{L}}(\Omega, f)$ are dense in Ω and we have

$$\dim_{\mathcal{H}} \mathcal{L}(\Omega, f) = \dim_{\mathcal{H}} \widetilde{\mathcal{L}}(\Omega, f) = \frac{d}{\min\{f(x) : x \in \Omega\}}. \tag{11}$$

The function f ranges over $[1,+\infty)$, since δ_x is always larger than 1.

As stated above, formula (7) shall now be seen as a particular case of (11). As in relation (7), (11) contains several results: the non-emptiness of $\widetilde{\mathcal{L}}(\Omega, f)$, the equality between the Hausdorff dimensions of $\mathcal{L}(\Omega, f)$ and $\widetilde{\mathcal{L}}(\Omega, f)$, and the value of this dimension.

The key point is that conditions $\mathcal{C}_1$ and $\mathcal{C}_2$ hold for many classical systems arising in ubiquity and number theory. In Section 5 we prove that Theorem 1.2 applies to the Diophantine approximation by dyadic numbers $\mathcal{D} = \Big((\mathbf{k}\cdot 2^{-j}, 2^{-j})\Big)_{j\geq 1, \mathbf{k}\in\{0,1,\dots,2^j-1\}^d}$, to the Diophantine approximation by rational

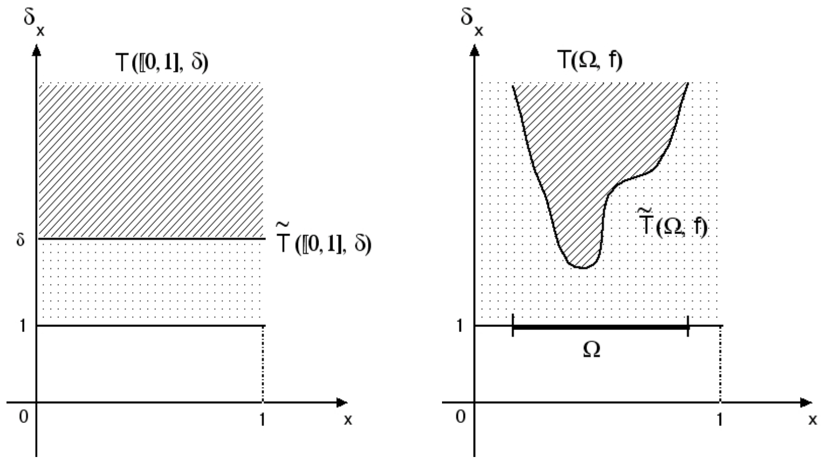

FIGURE 1. Geometric representation of $T(\Omega, f)$ and $\widetilde{T}(\Omega, f)$ for the constant function $f(x) = \delta$ on the left figure, for a typical continuous function f on the right figure.

numbers $\mathcal{R} = \big((p/q, 1/q^2)\big)_{q\geq 1,\, 0\leq p\leq q-1}$, to the so-called "inhomogeneous" Diophantine approximation by the system $\mathcal{I} = \Big(\big(\{n\alpha\}, \frac{1}{n}\big)\Big)_{n\geq 1}$, (where α is an irrational number whose approximation rate by the rational system $\mathcal{R}$ equals 2), and to the approximation rates by Poisson point processes $\mathcal{P}$.

Equality (11) can be interpreted geometrically. Consider the subsets of $\mathbb{R}^d \times \mathbb{R}$ defined by

$$\begin{aligned} T(\Omega, f) &= \{(x, \delta_x) : x \in \Omega \text{ and } \delta_x \geq f(x)\} \\ \widetilde{T}(\Omega, f) &= \{(x, \delta_x) : x \in \Omega \text{ and } \delta_x = f(x)\} \end{aligned}$$

Then $\mathcal{L}(\Omega, f)$ and $\widetilde{\mathcal{L}}(\Omega, f)$ are respectively the natural projections of $T(\Omega, f)$ and $\widetilde{T}(\Omega, f)$ on $\mathbb{R}^d$. Theorem 1.2 asserts that the "frontier" of $T(\Omega, f)$, $\widetilde{T}(\Omega, f)$, is non empty and that the projections of $T(\Omega, f)$ and $\widetilde{T}(\Omega, f)$ on $\mathbb{R}^d$ are both dense in Ω and have same Hausdorff dimension.

Changing our standpoint, Theorem 1.2 makes it possible to answer the following questions:

Are there real numbers $x \in [0, 1]$ satisfying $\delta_x = 1 + x$? $\quad \delta_x = 1/x$?

This question is of course not reachable via Jarnik's result, for which the approximation rate is a fixed number $\delta \geq 1$, independent of x. Moreover, it seems non-trivial (though possible) to explicitly construct an irrational number $x \in [0, 1]$ such that $\delta_x = 1 + x$. Theorem 1.2 implies for instance that, provided that a system $\mathcal{S}$ in $[0, 1]$ satisfies (8), $\mathcal{C}_1$ and $\mathcal{C}_2$, then

- for every real numbers $0 < a < b < 1$,

$$\dim_{\mathcal{H}} \Big\{ x \in [a,b] : \ \delta_x = 1 + x \Big\} = \frac{1}{1+a},$$

- for every real numbers $0 < a < b < 1$, for every $\alpha \geq 1$,

$$\dim_{\mathcal{H}} \Big\{ x \in [a,b] : \delta_x = \frac{\alpha}{x} \Big\} = \frac{b}{\alpha},$$

- and if $[a,b] \subset \left[\frac{1}{6}, \frac{5}{6}\right]$, then

$$\dim_{\mathcal{H}} \Big\{ x \in [a,b] : \ \delta_x = 2\sin(\pi x) \Big\} \ = \ \frac{1}{\min\big(2\sin(\pi a), 2\sin(\pi b)\big)}.$$

In the above equalities, the dimensions depend on the range of x. This was expected, since the conditions we impose on x depend on the non-constant continuous function f.

In order to prove Theorem 1.1 and the equality between the Hausdorff dimensions of the (classical) sets $\mathcal{L}_\delta$ and $\widetilde{\mathcal{L}}_\delta$ defined respectively in (2) and (3), the usual method consists in constructing iteratively a Borel probability measure m_δ of Hausdorff dimension larger than or equal to d/δ supported by $\mathcal{L}_\delta$. Then, recalling that $\dim_{\mathcal{H}} \widetilde{\mathcal{L}}_{\delta'} < d/\delta$ when $\delta' > \delta$, we deduce from (4) that $m_\delta(\widetilde{\mathcal{L}}_\delta) = 1$. Hence m_δ is supported by $\widetilde{\mathcal{L}}_\delta$ and $\dim_{\mathcal{H}}(\widetilde{\mathcal{L}}_\delta) = d/\delta$. Moreover, m_δ can be chosen as the Hausdorff measure associated with a suitable gauge function g satisfying $\lim_{r\to 0^+} \log(g(r))/\log(r) = 1/\delta$.

As shall be explained soon, this approach is inappropriate in the context of Theorem 1.2. First, observe that $\mathcal{L}(\Omega, f)$ and $\widetilde{\mathcal{L}}(\Omega, f)$ cannot be written as limsup sets. Nevertheless we still need to construct probability measures with support contained in $\widetilde{\mathcal{L}}(\Omega, f)$. This set is dense in Ω (like $\widetilde{\mathcal{L}}_\delta$ in the introduction), but in general it is mostly localized around those elements of Ω at which f reaches its minimum. This induces that in general, if B is a non-trivial closed ball inside Ω, then we have $\dim_{\mathcal{H}} (\widetilde{\mathcal{L}}(\Omega, f) \cap B) = \dim_{\mathcal{H}} \widetilde{\mathcal{L}}(\Omega, f)$ only if f reaches its minimum over B. This constitutes a notable difference with the sets $\widetilde{\mathcal{L}}_\delta$, for which $\dim_{\mathcal{H}} (\widetilde{\mathcal{L}}_\delta \cap B) = \dim_{\mathcal{H}} \widetilde{\mathcal{L}}_\delta$ for any non-trivial closed ball $B \subset [0,1]^d$. In particular, in general there is no Hausdorff measure whose restriction to $\widetilde{\mathcal{L}}(\Omega, f)$ is positive.

Let us illustrate our purpose.

In $[0,1]$, consider the system $\mathcal{R}$ associated with the rational numbers and the function $f(x) = 1 + x$ (the crucial property is the strict monotonicity of f). We are interested in $\mathcal{L}([0,1], f) = \{x \in [0,1] : \delta_x \geq 1 + x\}$ and $\widetilde{\mathcal{L}}([0,1], f) = \{x \in [0,1] : \delta_x = 1 + x\}$.

Jarnik's theorem obviously implies that $\dim_{\mathcal{H}} \mathcal{L}([0,1], f) = 1$. Indeed, using that $1 + x$ tends to 1 when $x > 0$ tends to 0, for every $\varepsilon > 0$, the set $\mathcal{L}([0,1], f) \cap [0, \varepsilon]$ contains all the real numbers whose approximation rate δ_x is larger than $1 + \varepsilon$. These real numbers form a set of Hausdorff dimension $1/(1 + \varepsilon)$. Letting ε tend to zero yields the result.

Similar arguments imply that for every $\varepsilon > 0$, $\widetilde{\mathcal{L}}([0,1], f) \cap [\varepsilon, 1]$ has Hausdorff dimension less than $1/(1+\varepsilon)$. However, Theorem 1.2 claims that the Hausdorff dimension of $\widetilde{\mathcal{L}}([0,1], f)$ is 1. Consequently, the elements of $\widetilde{\mathcal{L}}([0,1], f)$ responsible for the value of the Hausdorff dimension of $\widetilde{\mathcal{L}}([0,1], f)$ are "localized" around 0. Observe that in this context f does not reach its infimum.

For this reason, we refer to Theorem 1.2 as a localized Diophantine approximation. The proof will consist in constructing a family of Cantor sets $(\mathcal{K}_\varepsilon)_{\varepsilon>0}$, all included in $\widetilde{\mathcal{L}}([0,1]^d, f)$, which will be located closer and closer to one infimum of the function f. These Cantor sets will contain elements x with prescribed approximation rates (which may depend on x). The sequence of dimensions $\dim_{\mathcal{H}} \mathcal{K}_\varepsilon$ will be increasing to the desired dimension $\dfrac{d}{\min\{f(x) : x \in \Omega\}}$, as ε tends to zero.

We will prove Theorem 1.3, which is slightly more general than Theorem 1.2. This second version is determinant for its application to the analysis of the Hölder singularities of some Markov processes [2]. This extension addresses functions f which are continuous outside a set E with a given Hausdorff dimension.

Theorem 1.3. *Suppose that the assumptions of Theorem 1.2 on the system $\mathcal{S}$ are satisfied: (8), $\mathcal{C}_1$ and $\mathcal{C}_2$ hold for the system $\mathcal{S}$.*

Let Ω be a non-empty compact subset of $(0,1)^d$, such that $\overline{\mathring{\Omega}} = \Omega$.

Let $E \subset \Omega$ be a subset of Ω.

Let $f : (0,1)^d \to [1, \infty)$ be a function whose restriction to $\Omega \setminus E$ is continuous.

Suppose that $\dim_{\mathcal{H}} E < \dfrac{d}{\inf\{f(x) : x \in \Omega \setminus E\}}$. Then

$$\dim_{\mathcal{H}} \mathcal{L}(\Omega \setminus E, f) = \dim_{\mathcal{H}} \widetilde{\mathcal{L}}(\Omega \setminus E, f) = \frac{d}{\inf\{f(x) : x \in \Omega \setminus E\}}. \tag{12}$$

If, moreover, $\dim_{\mathcal{H}} E < \dfrac{d}{\sup\{f(x) : x \in \Omega \setminus E\}}$, then the sets $\mathcal{L}(\Omega \setminus E, f)$ and $\widetilde{\mathcal{L}}(\Omega \setminus E, f)$ are dense in Ω.

In general the sets $\mathcal{L}(\Omega \setminus E, f)$ and $\widetilde{\mathcal{L}}(\Omega \setminus E, f)$ cannot be studied by Khintchine-like formulas or by mass transference formulas as stated in [7, 9] (unless a localized version of these theories is developed). Moreover, they do not possess any large intersection properties [18], due to the presence of the non-constant function f.

The paper is organized as follows. Conditions $\mathcal{C}_1$ and $\mathcal{C}_2$, as well as some preliminary results, are given in Section 2. The lower bound in the two-sided equality (12) is proved in Section 3, while the corresponding upper bound is demonstrated in Section 4. Finally, several examples of suitable systems (including the rational system) are studied in Section 5.

2. Definitions and conditions $\mathcal{C}_1$ and $\mathcal{C}_2$

In $\mathbb{R}^d$, we work with the L^∞ norm.

2.1. Hausdorff measure, gauge functions and Hausdorff dimension.

Let ζ be a *gauge* function, i.e. a non-negative non-decreasing function on $\mathbb{R}_+$ such that $\lim_{x\to 0^+}\zeta(x)=0$. Let S be a subset of $\mathbb{R}^d$. For all $\eta>0$, let us define the quantity

$$\mathcal{H}^\zeta_\eta(S)=\inf\sum_{i\in\mathcal{I}}\zeta\left(|C_i|\right),$$

the infimum being taken over all the countable families $\{C_i\}_{i\in\mathcal{I}}$ of subsets of $\mathbb{R}^d$ such that $\bigcup_{i\in I}C_i$ is a covering of S and $|C_i|\le\eta$ for all $i\in I$ ($|C_i|$ stands for the diameter of C_i). As η decreases to 0, $\mathcal{H}^\zeta_\eta(S)$ is non-decreasing, and $\mathcal{H}^\zeta(S)=\lim_{\eta\to 0}\mathcal{H}^\zeta_\eta(S)$ defines an outer measure on $\mathbb{R}^d$, called Hausdorff ζ-measure, whose restriction to the Borel σ-field is a measure.

Let $\alpha>0$ be a real number. The α-dimensional Hausdorff measure on $\mathbb{R}^d$ is the measure $\mathcal{H}^{\zeta_\alpha}$, where $\zeta_\alpha(x)=x^\alpha$. For each $S\in\mathbb{R}^d$, there exists a unique real number $0\le D\le d$ such that $D=\sup\left\{\alpha>0:\mathcal{H}^{\zeta_\alpha}(S)=+\infty\right\}=\inf\left\{\alpha\ge 0:\mathcal{H}^{\zeta_\alpha}(S)=0\right\}$ (with the convention $\sup\emptyset=0$). This real number D is called the Hausdorff dimension of S and denoted $\dim_{\mathcal{H}}S$. We refer the reader to [17, 32] for more details.

If m is a Borel probability measure over $[0,1]^d$, then its lower and upper Hausdorff dimensions are respectively defined by

$$\begin{aligned}\dim_{\mathcal{H}*}(m) &= \inf\{\dim_{\mathcal{H}}B : m(B)>0\}\\ \dim_{\mathcal{H}}{}^*(m) &= \inf\{\dim_{\mathcal{H}}B : m(B)=1\}.\end{aligned}$$

When $\dim_{\mathcal{H}*}(m)=\dim_{\mathcal{H}}{}^*(m)$, this common value is called the Hausdorff dimension of m and denoted $\dim_{\mathcal{H}}(m)$.

2.2. Notations.

In the rest of the paper, we consider $(x_n)_{n\ge 1}$ a sequence of elements of $[0,1]^d$, and $(r_n)_{n\ge 1}$ a non-increasing sequence of positive real numbers converging to 0 when n tends to infinity. We then define the *system* $\mathcal{S}$ as the sequence of couples $\big((x_n,r_n)\big)_{n\ge 1}$.

For every integer $j\ge 0$, we denote by $\mathcal{G}_j$ the set of dyadic sub-cubes of $[0,1]^d$ of generation j, and let $\mathcal{G}_*$ stand for $\bigcup_{j\ge 1}\mathcal{G}_j$. For any dyadic cube $I\in\mathcal{G}_*$, we set $g(I)=-\log_2(|I|)$, the dyadic generation of I (recall that $|I|$ stands for the diameter of I).

We denote by Φ the set of functions $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ satisfying

- φ is a non-decreasing continuous functions such that $\varphi(0)=0$,
- $r\mapsto r^{-\varphi(r)}$ is decreasing and tends to infinity as $x>0$ tends to 0,
- for all real numbers $\alpha,\beta>0$, the mapping $r\mapsto r^{\alpha-\beta\varphi(r)}$ is increasing in a neighborhood of 0.

We introduce now the conditions on the system $\mathcal{S}$. These conditions essentially ensure an homogeneous repartition in $[0,1]^d$ of the points $(x_n)_{n\geq 1}$, and limit the overlaps between the balls $B(x_n, r_n)$.

2.3. **Condition $\mathcal{C}_1$: Weak redundancy.**

Definition 2.1. *Given the system $\mathcal{S} = \big((x_n, r_n)\big)_{n\geq 1}$, we define the irreducible sub-system $\big((y_n, \rho_n)\big)_{n\geq 1}$ associated with $\big((x_n, r_n)\big)_{n\geq 1}$ as follows:*

$$\big((y_n, \rho_n)\big)_{n\geq 1} = \big((x_n, r_n)\big)_{n\geq 1,\ n=\min\{p\geq 1:\, x_p = x_n\}}.$$

If $x \in \{x_n : n \geq 1\}$, then the irreducible subsystem $\big((y_n, \rho_n)\big)_{n\geq 1}$ contains one (and only one) couple of the form (x, r), where $r = \max\{r_n : (x_n, r_n) \in \mathcal{S}\}$. This definition is needed since the initial system $\big((x_n, r_n)\big)_{n\geq 1}$ may be very redundant (this occurs when one element x appears infinitely many times in the sequence $(x_n)_{n\geq 1}$, as in the case of the system of rational numbers $\big((p/q, 1/q^2)\big)_{q\geq 1,\, 0\leq p\leq q-1}$.)

Definition 2.2. *Let $\big((x_n, r_n)\big)_{n\geq 1}$ be a system, and consider its irreducible subsystem $\big((y_n, \rho_n)\big)_{n\geq 1}$. For any integer $j \geq 0$ we set*

$$\mathcal{T}_j = \big\{n : \quad 2^{-(j+1)} < \rho_n \leq 2^{-j}\big\}. \tag{13}$$

Condition $\mathcal{C}_1$: The system $\big((x_n, r_n)\big)_{n\geq 1}$ satisfies $\mathcal{C}_1$ when there exists a non-decreasing sequence of integers $(N_j)_{j\geq 0}$ such that

(1) we have $\displaystyle\lim_{j\to\infty} \frac{\log_2 N_j}{j} = 0$.
(2) for every $j \geq 1$, $\mathcal{T}_j$ can be decomposed into at most N_j pairwise disjoint subsets (denoted $\mathcal{T}_{j,1}, \ldots, \mathcal{T}_{j,N_j}$) such that for each $1 \leq i \leq N_j$, the balls $B(y_n, \rho_n)$, where n ranges over $\mathcal{T}_{j,i}$, are pairwise disjoint.

Each $\mathcal{T}_{j,i}$ has cardinality less than $2^{d(j+1)}$, and $\mathcal{T}_j$ has cardinality less than $N_j \cdot 2^{d(j+1)}$.

Condition $\mathcal{C}_1$ ensures that every $t \in [0,1]^d$ is covered by at most N_j balls of the form $B(y_n, \rho_n)$, $n \in \mathcal{T}_j$. The fact that N_j does not increase too fast toward infinity explains the appellation "weak redundancy" given to $\mathcal{C}_1$ in [3].

2.4. **Condition $\mathcal{C}_2$: a fine non-overlapping condition.**

In order to obtain Theorems 1.2 and 1.3, an additional property is required on the system. We emphasize that $\mathcal{C}_2$, though technical, is satisfied by many natural systems, as explained in Section 5. It appears that, except for the Poisson system, $\mathcal{C}_1$ and $\mathcal{C}_2$ are quite easy to check.

Definition 2.3. *Suppose that $\mathcal{S} = \big((x_n, r_n)\big)_{n\geq 1}$ satisfies $\mathcal{C}_1$, and consider the sequence $(N_j)_{j\geq 1}$ associated with $\mathcal{S}$ by $\mathcal{C}_1$.*

There exists a continuous function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $\psi(0) = 0$ *and for every* $j \geq 1$, N_j *can be written as*

$$N_j = 2^{dj\psi(2^{-j})}. \tag{14}$$

For every $\varphi \in \Phi$ *and for every* $j \geq 1$, *we define*

$$\begin{aligned} \gamma(j) &= \max\left\{k \in \mathbb{N} : N_k 2^{dk} \leq 2^{-dj\varphi(2^{-j})} 2^{dj}\right\} \\ &= \max\left\{k \in \mathbb{N} : 2^{dk(1+\psi(2^{-k}))} \leq 2^{dj(1-\varphi(2^{-j}))}\right\}. \end{aligned} \tag{15}$$

Obviously $\gamma(j) \leq j$, and the difference $j - \gamma(j)$ can be written as $j\theta(2^{-j})$, where the mapping $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and $\theta(0) = 0$.

The sequences $(\gamma(j))_{j\geq 1}$ and $(\theta(2^{-j}))_{j\geq 1}$ depend on the sequence $(N_j)_{j\geq 1}$ and on φ. Nevertheless, in the following, we omit to write this dependence, since by Property $\mathcal{C}_2$, both $(N_j)_{j\geq 1}$ and φ will be fixed once for all.

Definition 2.4. *Let* $\varphi \in \Phi$ *and* $(N_j)_{j\geq 1}$ *be defined as in Definition 2.3. Let* $V \in \mathcal{G}_*$ *be a dyadic cube in* $[0,1]^d$. *Let* $\delta > 1$ *be a real number.*

Recall that $g(V) = -\log_2 |V|$ *if the dyadic generation of* V.

The property $\mathcal{P}(V, \delta)$ *is said to hold when there exists* $x(V) \in V \subset [0,1]^d$ *and a positive real number* $r(V)$ *verifying:*

- $(x(V), r(V)) \in \mathcal{S}$,
- $2^{-g(V)-1} \leq r(V) < 2^{-g(V)}$,
- *and*

$$B\big(x(V), r(V)^\delta\big) \bigcap \left\{x_p : \gamma\big(g(V)\big) \leq -\log_2 r_p < [\delta(g(V)+1)] + 4\right\} = \{x(V)\}.$$

The notation $[y]$ stands for the integer part of the real number y.

Recall that $\gamma\big(g(V)\big) \leq g(V)$, and note that $[\delta g(V)]$ is heuristically the generation of the largest dyadic cube included in the contracted ball $B\left(x(V), r(V)^\delta\right)$.

$\mathcal{P}(V, \delta)$ holds when, except $x(V)$, all the elements x_p, where p ranges over the indices such that $\gamma\big(g(V)\big) \leq -\log_2 r_p < [\delta(g(V)+1)] + 4$, avoids the contracted ball $B\left(x(V), r(V)^\delta\right)$ (see Figure 2.). The constant 4 is due to technicalities along the proof. Note that $\mathcal{P}(V, \delta)$ depends on $(N_j)_{j\geq 1}$ and φ via γ (formula (15)), but as said above we do not mention this dependence since $(N_j)_{j\geq 1}$ and φ are fixed by $\mathcal{C}_2$.

$\mathcal{P}(V, \delta)$ seems to be a reasonable property, maybe not for all dyadic cubes V, but at least for a large number among them. Condition $\mathcal{C}_2$ is meant to ensure the validity of $\mathcal{P}(V, \delta)$ for a sufficient set of cubes V and approximation rates δ.

Condition $\mathcal{C}_2$: A system $\mathcal{S}$ satisfies $\mathcal{C}_2$ when $\mathcal{S}$ satisfies $\mathcal{C}_1$ and when there exists :

- a function $\varphi \in \Phi$,
- a non-decreasing sequence of integers $(N_j)_{j\geq 1}$ as in Definition 2.3,

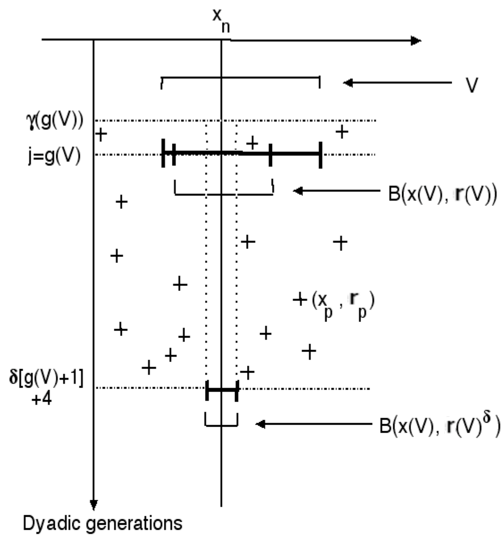

FIGURE 2. Property $\mathcal{P}(V,\delta)$

- a continuous function $\kappa : (1,+\infty) \to (0,1]$,
- a dense subset Δ of $(1,\infty)$,

with the following property:
For every $\delta \in \Delta$, for every dyadic cube U of $[0,1]^d$, there are infinitely many integers $j \geq g(U)$ satisfying

$$\#\mathcal{Q}(U,j,\delta) \ \geq \ \kappa(\delta) \cdot 2^{d(j-g(U))}, \tag{16}$$

where

$$\mathcal{Q}(U,j,\delta) = \left\{ V \in \mathcal{G}_j : \ V \subset U \quad \text{and} \quad \mathcal{P}\big(V,\delta\big) \text{ holds} \right\}.$$

In the following, the system $\mathcal{S}$ satisfies ($\mathcal{C}_1$ and) $\mathcal{C}_2$. Hence, φ and $(N_j)_{j\geq 1}$ are given, and all the parameters introduced from now on depend on them.

Observe that $2^{d(j-g(U))}$ is the number of dyadic cubes V of generation $j \geq g(U)$ included in U. Among them, $\mathcal{Q}(U,j,\delta)$ contains the cubes enjoying the property $\mathcal{P}\big(V,\delta\big)$. As claimed above, Condition $\mathcal{C}_2$ guarantees that given a dyadic cube U and $\delta \in \Delta$, infinitely often a given proportion of the dyadic subcubes V of generation j included in U satisfies $\mathcal{P}\big(V,\delta\big)$.

Remark 2.5. *For the rational system and other deterministic systems provided in Section 5, the function κ can be taken constant: $\forall\, \delta > 1$, $\kappa(\delta) = \kappa \in (0,1)$. The possible dependence in δ of the factor κ is introduced to include the systems obtained as Poisson point processes in the upper-half plane. This is explained in Section 5.*

2.5. **A preliminary result.** We shall need the following lemma, which requires only $\mathcal{C}_1$.

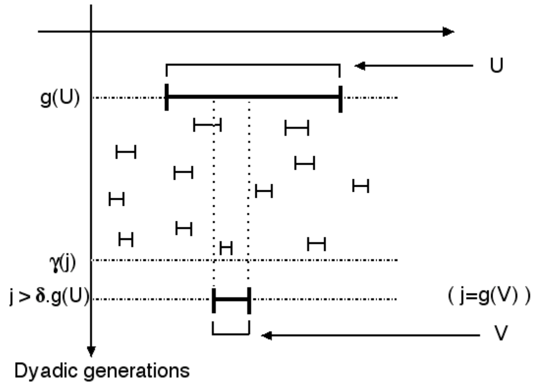

FIGURE 3. Property $\widetilde{\mathcal{Q}}(U,j,\delta)$

Lemma 2.6. *Let $\big((x_n,r_n)\big)_{n\geq 1}$ be a system and let $\big((y_n,\rho_n)\big)_{n\geq 1}$ be the corresponding irreducible subsystem. Suppose that $\mathcal{C}_1$ is satisfied.*

For every $\delta>1$, for every dyadic cube $U\in\mathcal{G}_*$, and every integer $j\geq \delta\cdot g(U)$, let us introduce the set of cubes

$$\widetilde{\mathcal{Q}}(U,j,\delta)=\left\{V\in\mathcal{G}_j : V\subset U,\, V\bigcap\left(\bigcup_{k=g(U)}^{\gamma(j)}\ \bigcup_{p\in\mathcal{T}_k} B(y_p,(\rho_p)^\delta)\right)\neq\emptyset\right\}.$$

Then, there exists a constant C_d depending only on d such that

$$\#\widetilde{\mathcal{Q}}(U,j,\delta)\leq C_d\cdot 2^{d(j-g(U))}\left[2^{-dj\varphi(2^{-j})}+\sum_{g(U)\leq k\leq j/\delta}2^{-dk(\delta-1-\psi(2^{-k}))}\right]. \tag{17}$$

The sets $\widetilde{\mathcal{Q}}(U,j,\delta)$ contains the dyadic cubes of generation j which intersect the irreducible balls $B(y_n,\rho_n)$ when n ranges in $\mathcal{T}_p$, $p\in[g(U),g(U)+1,...,\gamma(j)]$. It is crucial in the further construction of Cantor sets that $\widetilde{\mathcal{Q}}(U,j,\delta)$ cannot contain a too large number of cubes (see Figure 3.).

Our proof will use the following standard estimates.

Lemma 2.7. *There exists a constant C'_d depending on d only such that:*

1. *If $r_0>0$, $j\in\mathbb{N}$ and B is a closed ball such that $2^{-j}\leq|B|\leq r_0$, then B intersects at most $C'_d\cdot 2^{dj}\cdot r_0$ elements of $\mathcal{G}_j$.*
2. *If $U\in\mathcal{G}_*$, k is an integer larger than $g(U)$ and $\mathcal{T}$ is a family of pairwise disjoint closed balls of radius larger than $2^{-(k+1)}$, then U intersects at most $C'_d\cdot 2^{d(k-g(U))}$ elements of $\mathcal{T}$.*

Proof of Lemma 2.6. Let $U\in\mathcal{G}_*$ and $j\geq\delta\cdot g(U)$. Let k be an integer such that $k\in[g(U),\gamma(j)]$.

We are going to count the number of dyadic cubes V in $\mathcal{G}_j$ which are included in U and which intersect balls of the form $B(y_p,(\rho_p)^\delta)$ for some $p\in\mathcal{T}_k$. Two cases shall be distinguished:

• If $g(U) \leq k \leq j/\delta$, then for $p \in \mathcal{T}_k$ we have $2^{-j} \leq 2^{-k\delta} \leq |B(y_p, (\rho_p)^\delta)| \leq 2^{1-k\delta}$. Consequently, $B(y_p, (\rho_p)^\delta)$ intersects at most $C'_d \cdot 2^{dj} 2^{d(1-k\delta)}$ elements of $\mathcal{G}_j$.

Moreover, by construction, $\mathcal{T}_k = \bigcup_{1\leq l \leq N_k} \mathcal{T}_{k,l}$, where the elements of each $\mathcal{T}_{k,l}$ are pairwise disjoint closed balls of radius larger than $2^{-(k+1)}$. Consequently, for $1 \leq l \leq N_k$, the cardinality of those integers $p \in \mathcal{T}_{k,l}$ satisfying $B(y_p, \rho_p) \cap U \neq \emptyset$ is at most $C'_d \cdot 2^{d(k-g(U))}$. Thus, there are at most $C'_d \cdot N_k 2^{d(k-g(U))}$ integers $p \in \mathcal{T}_k$ satisfying $B(y_p, \rho_p) \cap U \neq \emptyset$.

Combining the last remarks, the cardinality of the subset of $\mathcal{G}_j$ whose elements are included in U and meet a ball $B(y_p, (\rho_p)^\delta)$ with $p \in \mathcal{T}_k$, is less than

$$(C'_d)^2 N_k 2^{d(k-g(U))} \cdot 2^{dj} 2^{d(1-k\delta)} \quad = \quad 2^d (C'_d)^2 \cdot 2^{d(j-g(U))} N_k \cdot 2^{-dk(\delta-1)}.$$

• If $j/\delta < k \leq \gamma(j)$, then for every $p \in \mathcal{T}_k$ we have $|B(y_p, (\rho_p)^\delta)| \leq 2^{1-j}$. Hence $B(y_p, (\rho_p)^\delta)$ intersects at most 3^d cubes of $\mathcal{G}_j$. Consequently, the cardinality of the subset of $\mathcal{G}_j$ whose elements are included in U and meet a ball $B(y_p, (\rho_p)^\delta)$ with $p \in \mathcal{T}_k$ is at most $3^d C'_d \cdot N_k 2^{d(k-g(U))}$.

Summarizing the above estimates, we obtain

$$\begin{aligned} \#\widetilde{\mathcal{Q}}(U, j, \delta) \quad \leq \quad & 2^d (C'_d)^2 \cdot 2^{d(j-g(U))} \sum_{g(U) \leq k \leq j/\delta} N_k 2^{-kd(\delta-1)} \\ & + 3^d C'_d \cdot 2^{-dg(U)} \sum_{j/\delta < k \leq \gamma(j)} N_k 2^{dk}. \end{aligned}$$

Using that $(N_k)_{k\geq 1}$ is non-decreasing, we get

$$\begin{aligned} 3^d C'_d \cdot 2^{-dg(U)} \sum_{j/\delta < k \leq \gamma(j)]} N_k 2^{dk} \quad & \leq \quad 3^d C'_d \cdot 2^{-dg(U)} N_{\gamma(j)} \sum_{j/\delta < k \leq \gamma(j)} 2^{dk} \\ & \leq \quad 2 \cdot 3^d C'_d \cdot N_{\gamma(j)} \, 2^{d\gamma(j)} \, 2^{-dg(U)} \\ & \leq \quad 2 \cdot 3^d C'_d \cdot 2^{d(j-g(U))} 2^{-dj\varphi(2^{-j})}, \end{aligned}$$

where we used the definition of $\gamma(j)$ in the last inequality. Moreover, using the definition (14) of $\psi(2^{-k})$ based on N_k, we find that

$$2^d (C'_d)^2 \cdot 2^{d(j-g(U))} N_k 2^{-kd(\delta-1)} \leq 2^d (C'_d)^2 \cdot 2^{d(j-g(U))} 2^{-kd(\delta-1-\psi(2^{-k}))}.$$

Equation (17) follows easily.

3. Lower bound for the Hausdorff dimensions in Theorem 1.3

Let $\mathcal{S} = \big((x_n, r_n)\big)_{n\geq 1}$ be a system satisfying $\mathcal{C}_1$, and let φ, $(N_j)_{j\geq 1}$, and Δ be fixed so that $\mathcal{S}$ satisfies also $\mathcal{C}_2$. We denote by $\big((y_n, \rho_n)\big)_{n\geq 1}$ the irreducible subsystem of $\mathcal{S}$.

Consider Ω, E and f as in Theorem 1.3. We have

$$h := \frac{d}{\inf\{f(x) : x \in \Omega \setminus E\}} > \dim_{\mathcal{H}} E. \tag{18}$$

Our aim is to prove that the Hausdorff dimension of $\widetilde{\mathcal{L}}(\Omega \setminus E, f) = \{x \in \Omega \setminus E : \delta_x = f(x)\}$ equals h. We are going to construct a family of Cantor sets all included in $\widetilde{\mathcal{L}}(\Omega \setminus E, f)$ and such that the supremum of their Hausdorff dimensions is larger than (or equal to) h.

3.1. **First simplifications.** Before starting the constructions, we make some remarks:

- If the restriction of f to $\Omega \setminus E$ is equal to the minimum of f over $U \cap \overset{\circ}{\Omega} \setminus E$, where U is a non-empty dyadic cube, then the result follows from Theorem 1.1. Thus we will assume that this is not the case, i.e. the subset of $\Omega \setminus E$ over which f reaches its minimum is nowhere dense (this set is empty in general).
- Problems may occur in the construction below when $h = d$, i.e. when $\inf\{f(x) : x \in \Omega \setminus E\} = 1$. Let us explain how we circumvent such difficulties. Assume that $\inf\{f(x) : x \in \Omega \setminus E\} = 1$. For every $\varepsilon > 0$ small enough, it is possible to find a dyadic cube U_ε such that the restriction of f to $U_\varepsilon \cap (\Omega \setminus E)$ has an infimum which belongs to the open interval $(1, 1 + \varepsilon)$. The construction below can be applied to the set $\widetilde{\mathcal{L}}(U_\varepsilon \cap (\Omega \setminus E), f)$, and we find that $\dim_{\mathcal{H}} \widetilde{\mathcal{L}}(U_\varepsilon \cap (\Omega \setminus E), f) \geq d/(1 + \varepsilon)$. Letting ε tend to zero yields the result.

Thus we assume that h defined by (18) is strictly less than d.

3.2. **Preliminary work.** Fix $\varepsilon \in (0, h)$, and recall that $h < d$. By definition (18) of h, and since $\dim_{\mathcal{H}} E < d = \dim_{\mathcal{H}} \Omega$, there exists $y_\varepsilon \in \overset{\circ}{\Omega} \setminus E$ such that $h - \varepsilon/2 \leq \dfrac{d}{f(y_\varepsilon)} \leq h$. Hence, using the continuity of f at y_ε, for every y in a neighborhood $\Omega_\varepsilon \subset \overset{\circ}{\Omega}$ small enough around y_ε, we have $h - \varepsilon \leq \dfrac{d}{f(y)} \leq h$. Equivalently, when ε small enough, we have

$$\forall\, y \in \Omega_\varepsilon, \quad \frac{d}{h} \leq f(y) \leq \frac{d}{h - \varepsilon} \leq \frac{d}{h}\left(1 + 2\frac{\varepsilon}{h}\right). \tag{19}$$

Recall that Δ is the set of admissible approximation rates allowed by property $\mathcal{C}_2$. In every dyadic cube $V \in \mathcal{G}_*$ included in Ω_ε, we pick up an element $y_V \in V$ and we choose a real number $\delta(V) \in \Delta$ such that (φ is fixed by $\mathcal{C}_2$ and ψ is defined by (14))

$$\delta(V) \in \big[f(y_V) + d(\varphi(|V|) + \psi(|V|)), f(y_V) + 3d(\varphi(|V|) + \psi(|V|))\big]. \tag{20}$$

Observe that the real numbers $\delta(V)$ are bounded from above and below, since ϕ and ψ are continuous and f is bounded on Ω_ε. Moreover, by formula

(19) there exists a constant $\alpha > 1$ such that for every V having diameter small enough one has

$$\delta(V) - 3d\varphi(|V|) \geq \alpha > 1. \tag{21}$$

Since the function $\kappa(\cdot)$ determined by condition $\mathcal{C}_2$ is continuous, there is a constant $\kappa \in (0,1)$ such that for every δ belonging to the set $\{\delta(V) : V \text{ dyadic cube } \subset \Omega_\varepsilon\}$, for every dyadic cube $U \subset \Omega_\varepsilon$, (16) holds infinitely often with the same constant κ (instead of $\kappa(\delta)$). We choose κ so that $2^{d+1}/\kappa$ is a positive power of 2. This will simplify a little bit the forthcoming constructions.

We now start the construction of a Cantor set $\mathscr{K}_\varepsilon$ such that $\mathscr{K}_\varepsilon \setminus E$ is included in $\widetilde{\mathcal{L}}(\Omega \setminus E, f) \cap \Omega_\varepsilon$ and simultaneously a probability measure μ_ε supported by $\mathscr{K}_\varepsilon$ such that $\dim_{\mathcal{H}*}(\mu_\varepsilon) \geq h_\varepsilon$, for some real number h_ε which satisfies $\lim_{\varepsilon \to 0^+} h_\varepsilon = h$.

Assume for a while that the construction of μ_ε and $\mathscr{K}_\varepsilon$ is achieved. Then the lower bound in (12) of Theorem 1.3 is obtained by the following argument. Since $\dim_{\mathcal{H}} E < h$, when ε is small enough we have $\mu_\varepsilon(E) = 0$. Recalling that the support of μ_ε is $\mathscr{K}_\varepsilon$, we deduce that

$$\dim_{\mathcal{H}} \widetilde{\mathcal{L}}(\Omega \setminus E, f) \geq \dim_{\mathcal{H}} \mathscr{K}_\varepsilon \setminus E \geq \dim_{\mathcal{H}}(\mu_\varepsilon) \geq h_\varepsilon.$$

Letting ε tend to 0 yields

$$\dim_{\mathcal{H}} \widetilde{\mathcal{L}}(\Omega \setminus E, f) \geq h.$$

The Cantor set $\mathscr{K}_\varepsilon$ will be obtained as a limsup set of the form

$$\mathscr{K}_\varepsilon = \bigcap_{n \geq 0} \bigcup_{U \in \mathcal{F}_n} U,$$

where for every $n \geq 0$, $\mathcal{F}_n$ is a collection of pairwise disjoint closed dyadic cubes U such that each element of $\mathcal{F}_{n+1}$ is included in one (and by construction only one) element of $\mathcal{F}_n$.

The sequence $(\mathcal{F}_n)_{n \geq 0}$ is built by induction, as follows.

At first, we choose a dyadic cube U_0 included in Ω_ε, small enough so that $3d(\varphi(|U_0|) + \psi(|U_0|)) \leq \varepsilon$ and $\dfrac{2^{d+1}}{\kappa} \leq |U_0|^{-d\varphi(|U_0|)}$, where we recall that κ is the constant appearing in (16) (the dependence on δ has been removed by an argument above). We define $\mathcal{F}_0 = \{U_0\}$. This choice also implies that for any dyadic cube $U \subset U_0 \subset \Omega_\varepsilon$ that we are going to consider, we have (using (19) and (20))

$$\delta(U) \leq f(x_\varepsilon) + \varepsilon \left(2\frac{d}{h^2} + 1\right) =: H_\varepsilon. \tag{22}$$

3.3. **Construction of the first generation of the Cantor set, $\mathcal{F}_1$.** Let us find the elements of $\mathcal{F}_1$.

We apply property $\mathcal{C}_2$ and Lemma 2.6 with $U = U_0$ and $\delta = \delta(U_0)$. This yields that there are infinitely many integers $j \geq g(U_0)$ such that

$$\begin{aligned} \#\mathcal{Q}(U_0, j, \delta(U_0)) &\geq \kappa \cdot 2^{d(j-g(U_0))} \\ \text{and } \#\widetilde{\mathcal{Q}}(U_0, j, \delta(U_0)) &\leq C_d \cdot 2^{d(j-g(U_0))} \Big(2^{-dj\varphi(2^{-j})} + \\ &\sum_{g(U_0) \leq k \leq j/\delta(U_0)} 2^{-dk(\delta(U_0)-1-\psi(2^{-k}))} \Big). \end{aligned}$$

We use (21) to bound from above the sum in the second equation:

$$\sum_{g(U_0) \leq k \leq j/\delta(U_0)} 2^{-dk(\delta(U_0)-1-\psi(2^{-k}))} \leq \sum_{g(U_0) \leq k} 2^{-dk\alpha} \leq C \cdot 2^{-d\alpha g(U_0)}.$$

The constant C does not depend on U_0. Consequently, the second upper bound above can be simplified into

$$\#\widetilde{\mathcal{Q}}(U_0, j, \delta(U_0)) \leq C_d \cdot C \cdot 2^{d(j-g(U_0))} (2^{-j\varphi(2^{-j})} + 2^{-d\alpha g(U_0)}).$$

Provided that U_0 has diameter small enough and j is large enough, we have $\#\widetilde{\mathcal{Q}}(U_0, j, \delta(U_0)) \leq \kappa/4 \cdot 2^{d(j-g(U_0))}$. From the inequalities between the cardinalities of $\mathcal{Q}(U_0, j, \delta(U_0))$ and $\widetilde{\mathcal{Q}}(U_0, j, \delta(U_0))$, we deduce that for j large enough, there is a subset $\widetilde{\mathcal{F}}_1$ of cardinality at least $\frac{\kappa}{2} \cdot 2^{d(j-g(U_0))}$ in $\mathcal{Q}(U_0, j, \delta(U_0)) \setminus \widetilde{\mathcal{Q}}(U_0, j, \delta(U_0))$. Moreover, we can find at least $\#\widetilde{\mathcal{F}}_1/2^d$ elements of $\widetilde{\mathcal{F}}_1$ which are distant from each other by at least 2^{-j}. Consequently, we can assume that there are exactly $\frac{\kappa}{2^{d+1}} \cdot 2^{d(j-g(U_0))}$ dyadic cubes in $\widetilde{\mathcal{F}}_1$, whose mutual distance is at least 2^{-j}.

By construction, each cube $\widetilde{V} \in \widetilde{\mathcal{F}}_1$ satisfies simultaneously $\widetilde{V} \in \mathcal{G}_j$, $\widetilde{V} \subset U_0$, $\mathcal{P}\big(\widetilde{V}, \delta(U_0)\big)$ and (recall that $j = g(\widetilde{V})$)

$$\widetilde{V} \bigcap \Big(\bigcup_{k=g(U_0)}^{\gamma(j)} \bigcup_{p \in \mathcal{T}_k} B(y_p, (\rho_p)^{\delta(U_0)}) \Big) = \emptyset.$$

Combining the information, each cube $\widetilde{V} \in \widetilde{\mathcal{F}}_1$ contains an element $x(\widetilde{V})$ such that $(x(\widetilde{V}), r(\widetilde{V})) \in \mathcal{S}$ for some radius $r(\widetilde{V})$ satisfying $2^{-j-1} \leq r(\widetilde{V}) < 2^{-j}$. By construction we have

$$\begin{aligned} &B\left(x(\widetilde{V}), r(\widetilde{V})^{\delta(U_0)} \right) \bigcap \left\{ x_p : \ \gamma(j) \leq -\log_2 r_p < (j+1)\delta(U_0) + 4 \right\} \\ &= \{x(\widetilde{V})\}. \end{aligned}$$

In order to compute the Hausdorff dimension of the limsup sets we are interested in, we must find points with a prescribed approximation rate in a very precise way. For this, a new definition is needed.

Definition 3.1. *For $\delta > 1$, $\varepsilon > 0$, $x \in \mathbb{R}^d$ and $r > 0$ we define the annulus*

$$A(x, r, \delta, \varepsilon) = B(x, r^\delta) \setminus B(x, r^{\delta+\varepsilon}).$$

For each $\widetilde{V} \in \widetilde{\mathcal{F}}_1$, consider the associated annulus

$$A(\widetilde{V}) = A(x(\widetilde{V}), r(\widetilde{V}), \delta(U_0), \varphi(2^{-j})).$$

Remark 3.2. *The diameter of $A(\widetilde{V})$ is $2 \cdot r(\widetilde{V})^{\delta(U_0)} = 2^{1+\log_2(r(\widetilde{V})^{\delta(U_0)})}$. Provided that j is taken large enough, the "hole" in the annulus $A(\widetilde{V})$ is extremely small, since the ratio $\frac{r(\widetilde{V})^{\delta(U_0)}}{r(\widetilde{V})^{\delta(U_0)+\varphi(2^{-j})}} = r(\widetilde{V})^{-\varphi(2^{-j})}$ tends to infinity when j tends to infinity (recall that φ belong to the functional space Φ and $r(\widetilde{V}) \sim 2^{-j}$).*

Let $\widetilde{\widetilde{V}}$ be one of the largest closed dyadic cubes included in $A(\widetilde{V}) \bigcap \widetilde{V}$. Using Remark 3.2, the generation $g(\widetilde{\widetilde{V}})$ of the dyadic cube $\widetilde{\widetilde{V}}$ is at most equal to $[-\log_2(r(\widetilde{V})^{\delta(U_0)})] + 3$.

We choose then V to be one of the subcubes of $\widetilde{\widetilde{V}}$ of generation $g(\widetilde{\widetilde{V}})+1$ among those cubes of this generation which are the closest to $x(\widetilde{V})$. We obtain that:

- the dyadic generation of V satisfies

$$[-\log_2(r(\widetilde{V})^{\delta(U_0)})] \leq g(V) \leq [-\log_2(r(\widetilde{V})^{\delta(U_0)})] + 4, \tag{23}$$

- for each p such that $\gamma(j) \leq -\log_2 r_p \leq [(j+1)\delta(U_0)] + 4$, if $x_p \neq x_n = x(\widetilde{V})$ then $x_p \notin B(x(\widetilde{V}), r(\widetilde{V})^{\delta(U_0)})$,
- for each p such that $\gamma(j) \leq -\log_2 r_p \leq [(j+1)\delta(U_0)] + 4$, for all $x \in V$ we have (using the function θ defined in Definition 2.3)

$$\begin{aligned} |x - x_p| \geq |V| \ &\geq \ r(\widetilde{V})^{\delta(U_0)}/16 \ = \ 2^{(\gamma(j)-j)\delta(U_0)} \cdot 2^{-\gamma(j)\delta(U_0)}/16 \\ &\geq \ 2^{-j\theta(2^{-j})\delta(U_0)} \cdot r_p^{\delta(U_0)}/16 \ \geq \ r_p^{\delta(U_0)+\theta(2^{-j})\delta(U_0)/2}/16. \end{aligned}$$

  The last inequality follows from the fact that, when j is large enough, $r_p \leq 2^{-\gamma(j)} \leq 2^{-j/2}$. Using (22), we see that

$$|x - x_p| \ \geq \ r_p^{\delta(U_0)+\theta(2^{-j})H_\varepsilon/2}/16.$$

When two dyadic cubes V and $\widetilde{V}$ are related via such a relationship, we say that V **is the contracted descendant of** $\widetilde{V}$.

The previous construction guarantees that (recall that $j = g(\widetilde{V})$):

- (23) holds,
- since $V \subset A(\widetilde{V})$, every element $x \in V$ is approximated at a rate $\in [\delta(U_0), \delta(U_0) + \varphi(2^{-j})]$ by $(x(\widetilde{V}), r(\widetilde{V}))$,
- for every $x \in V$, for every $k \in \{g(U_0), ..., \gamma(j)\}$, for every $p \in \mathcal{T}_k$, $x \notin B(x_p, (r_p)^{\delta(U_0)})$, i.e. x is not approximated at rate larger than $\delta(U_0)$ by these couples $(x_p, r_p) \in \mathcal{S}$,

- for every $x \in V$, for every integer p such that
$$\gamma(j) \leq [-\log_2 r_p] < [-\log_2(r(\widetilde{V})^{\delta(U_0)})] + 4,$$
we have that $x \notin B\big(x_p, (r_p)^{\delta(U_0)+\theta(2^{-j})H_\varepsilon/2}/16\big)$.
- The first, third and fourth previous items imply that if p is such that $2^{-g(V)} \leq r_p \leq 2^{-g(U_0)}$ and $x_p \neq x(\widetilde{V})$, then for all $x \in V$ we have $x \notin B\big(x_p, r_p^{\delta(U_0)+\theta(2^{-j})H_\varepsilon/2}/16\big)$.

Since this situation occurs for an infinite number of generations j, we choose j large enough so that

$$j \geq 2g(U_0) \quad \text{and} \quad \max\left(\frac{2^{d+1}}{\kappa}, 2^{dg(U_0)}\right) \leq 2^{dj\varphi(2^{-j})}. \tag{24}$$

The previous inequality ensures that $\frac{2^{d+1}}{\kappa} \leq |V|^{-d\varphi(|V|)}$. This will play a role in Section 3.6.

By construction, we have $|V| \geq 2^{-(j+1)\delta(U_0)}/16$. Consequently, using (22) (i.e. $\delta(U)$ is bounded above by H_ε independently of U), without loss of generality one can suppose that j is large enough so that for some constant $C > 0$ (depending on H_ε),

$$\text{for every } \widetilde{V} \in \mathcal{Q}(U_0, j, \delta(U_0)), \quad |V| \geq C \cdot 2^{-j\delta(U_0)}. \tag{25}$$

This yields a precise relationship between the diameter of a cube $\widetilde{V} \in \widetilde{\mathcal{F}}_1$ and the diameter of its contracted descendant V.

Now, let us consider the set of contracted descendants of the elements of $\widetilde{\mathcal{F}}_1$

$$\mathcal{F}_1 = \{V : V \text{ is the contracted descendant of some } \widetilde{V} \in \widetilde{\mathcal{F}}_1\}.$$

We construct a measure μ_ε on the algebra $\sigma_1 = \sigma(V : V \in \mathcal{F}_1)$ generated by the dyadic cubes of $\mathcal{F}_1$ by imposing:

$$\forall\, V \in \mathcal{F}_1, \quad \mu_\varepsilon(V) = (\#\mathcal{F}_1)^{-1}.$$

Let $V \in \mathcal{F}_1$. Recalling that $\#\widetilde{\mathcal{F}}_1 = \frac{\kappa}{2^{d+1}} \cdot 2^{d(j-g(U_0))}$, using (24) we get

$$\mu_\varepsilon(V) \leq \frac{2^{d+1}}{\kappa} \cdot 2^{d(g(U_0)-j)} \leq 2^{2dj\varphi(2^{-j})-dj}.$$

Using (25) we find that for some universal constant $C > 0$

$$\mu_\varepsilon(V) \leq C \cdot 2^{2dj\varphi(2^{-j})}|V|^{d/\delta(U_0)}.$$

Due to the monotonicity of $r^{-\varphi(r)}$, we have $2^{2dj\varphi(2^{-j})} \leq |V|^{-2d\varphi(|V|)}$, and when j is chosen large enough, $C \leq |V|^{-2d\varphi(|V|)}$. All these computations yield

$$\forall\ V \in \mathcal{F}_1, \quad \mu_\varepsilon(V) \leq |V|^{d/\delta(U_0)-3d\varphi(|V|)} \tag{26}$$

We now fix the integer $j = j_0$ so that all the assumptions above are satisfied. The last property of these cubes of first generation is that for every $V \neq V' \in \mathcal{F}_1$, the distance between V and V' is greater than 2^{-j_0}.

3.4. **Construction of $\mathcal{F}_2$, the second generation of the Cantor set.** The second generation is obtained as follows.

Note that, thanks to (24), we insured in the previous step that for each $U_1 \in \mathcal{F}_1$ we have $\frac{2^{d+1}}{\kappa} \leq |U_1|^{-d\varphi(|U_1|)}$.

Given $U_1 \in \mathcal{F}_1$, we know that there are infinitely many $j \geq g(U_1)$ such that

$$\begin{aligned} \#\mathcal{Q}(U_1, j, \delta(U_1)) &\geq \kappa \cdot 2^{d(j-g(U_1))} \\ \#\widetilde{\mathcal{Q}}(U_1, j, \delta(U_1)) &\leq C_\delta \cdot 2^{d(j-g(U_1))}\Big(2^{-dj\varphi(2^{-j})} \\ &\quad + \sum_{g(U_1)\leq k\leq j/\delta(U_1)} 2^{-dk(\delta(U_1)-1-\psi(2^{-k}))}\Big). \end{aligned}$$

The arguments used in the first step to find an upper bound for the sum in the second inequality above also apply here. When j is chosen large enough, we can find a subset $\widetilde{\mathcal{F}}_2(U_1)$ of cardinality $\dfrac{\kappa}{2^{d+1}} 2^{d(j-g(U_1))}$ in $\mathcal{Q}(U_1, j, \delta(U_1)) \setminus \widetilde{\mathcal{Q}}(U_1, j, \delta(U_1))$ such that

- the dyadic cubes $\widetilde{V}$ belonging to $\widetilde{\mathcal{F}}_2(U_1)$ are mutually distant from at least 2^{-j},
- each dyadic cube $\widetilde{V} \in \widetilde{\mathcal{F}}_2(U_1)$ satisfies simultaneously $\widetilde{V} \in \mathcal{G}_j$, $\widetilde{V} \subset U_1$, $\mathcal{P}\big(\widetilde{V}, \delta(U_1)\big)$ and

$$\widetilde{V} \bigcap \left(\bigcup_{k=g(U_1)}^{\gamma(j)} \bigcup_{p\in\mathcal{T}_k} B(y_p, (\rho_p)^{\delta(U_1)}) \right) = \emptyset.$$

As in the first step, we associate with every $\widetilde{V} \in \widetilde{\mathcal{F}}_2(U_1)$ a dyadic cube called its contracted descendant V, which enjoys the following properties:

- There exists an element $x(\widetilde{V}) \in \widetilde{V}$ and a positive real number $r(\widetilde{V})$ such that $(x(\widetilde{V}), r(\widetilde{V})) \in \mathcal{S}$, $r(\widetilde{V})$ satisfies $2^{-j-1} \leq r(\widetilde{V}) \leq 2^{-j}$, and every $x \in V$ is approximated at a rate belonging to $[\delta(U_1), \delta(U_1) + \varphi(2^{-j})]$ by $(x(\widetilde{V}), r(\widetilde{V}))$,
- if p is such that $2^{-g(V)} \leq r_p \leq 2^{-g(U_1)}$ and $x_p \neq x_n$, then for all $x \in V$ we have $x \notin B\big(x_p, (r_p)^{\delta(U_1)+\theta(2^{-j})H_\varepsilon/2}/16\big)$.

We now fix the integer $j = j(U_1)$ so that all the assumptions above are satisfied, and we set

$$\mathcal{F}_2(U_1) = \{V : V \text{ is the contracted descendant of one } \widetilde{V} \in \widetilde{\mathcal{F}}_2(U_1)\},$$

and

$$\mathcal{F}_2 = \{V \in \mathcal{G} : \ \exists U_1 \in \mathcal{F}_1 \text{ such that } V \in \mathcal{F}_2(U_1)\}.$$

The measure μ_ε can be extended into a Borel probability measure on the algebra $\sigma_2 = \sigma(L : L \in \mathcal{F}_1 \cup \mathcal{F}_2)$ by imposing

$$\text{for every } U_1 \in \mathcal{F}_1, \text{ for every } V \in \mathcal{F}_2(U_1), \quad \mu_\varepsilon(V) = \frac{\mu_\varepsilon(U_1)}{\#\mathcal{F}_2(U_1)}.$$

We choose $j_1 := \min(j(U_1) : U_1 \in \mathcal{F}_1)$ large enough so that for every $U_1 \in \mathcal{F}_1$, $j_1 \geq 2g(U_1)$ and

$$\max\left(\frac{2^{d+1}}{\kappa}, 2^{dg(U_1)}, |U_1|^{d/\delta(U_0)-3d\varphi(|U_1|)}\right) \leq 2^{dj_1\varphi(2^{-j_1})}. \tag{27}$$

In particular, for every $V \in \mathcal{F}_2(U_1)$ we have $\frac{2^{d+1}}{\kappa} \leq |V|^{-d\varphi(|V|)}$. Moreover, for some constant $C > 0$,

$$\text{for every } V \in \mathcal{F}_2(U_1), \quad |V| \geq C \cdot 2^{-dj(U_1)\delta(U_1)}. \tag{28}$$

Let us check the scaling properties of the measure μ_ε on the elements of σ_2. Let $U_1 \in \mathcal{F}_1$ and $V \in \mathcal{F}_2(U_1)$. Combining (26) and the lower bound for the cardinality of $\widetilde{\mathcal{F}}_2(U_1)$, we obtain that

$$\begin{aligned} \mu_\varepsilon(V) &\leq \frac{2^{d+1}}{\kappa} \cdot 2^{d(g(U_1)-j(U_1))}\mu_\varepsilon(U_1) \\ &\leq \frac{2^{d+1}}{\kappa} \cdot 2^{d(g(U_1)-j(U_1))}|U_1|^{d/\delta(U_0)-3d\varphi(|U_1|)}. \end{aligned}$$

By (27) and then (28), we get

$$\mu_\varepsilon(V) \leq 2^{-dj(U_1)}2^{-3dj_1\varphi(2^{-j_1})} \leq C|V|^{d/\delta(U_1)}2^{-3dj_1\varphi(2^{-j_1})}.$$

Using the monotonicity of $r \mapsto r^{-\varphi(r)}$, which tends to $+\infty$ when $r \to 0^+$, we see that $|V|^{-\varphi(|V|)} \geq 2^{-j_1\varphi(2^{-j_1})}$ when j_1 is large enough. We get

$$\mu_\varepsilon(V) \leq |V|^{d/\delta(U_1)-3d\varphi(|V|)}. \tag{29}$$

As in the first step, given $U_1 \in \mathcal{F}_1$, for any pair of distinct elements of $\mathcal{F}_2(U_1)$, namely (V, V'), we have $d(V, V') \leq 2^{-j(U_1)}$.

3.5. **Induction.** Suppose that for $n \geq 2$ we have constructed $\mathcal{F}_0, \dots, \mathcal{F}_n$, a finite sequence of sets of closed dyadic cubes, as well as a measure μ_ε on $\sigma_n = \sigma\Big(I : I \in \bigcup_{1\leq m\leq n} \mathcal{F}_m\Big)$ such that:

(1) For every $1 \leq m \leq n$, each element U of $\mathcal{F}_m$ is included in one element of $\mathcal{F}_{m-1}$, and satisfies $\dfrac{2^{d+1}}{\kappa} \leq |U|^{-d\varphi(|U|)}$.

(2) For every $1 \leq m \leq n$, if $U \in \mathcal{F}_{m-1}$, there exists a dyadic generation $j(U)$ such that:

(a) We have

$$2g(U) \leq j(U) \text{ and } \#\{V \in \mathcal{F}_m : V \subset U\} = \frac{\kappa \cdot 2^{d(j(U)-g(U))}}{2^{d+1}}, \tag{30}$$

and if two distinct elements V and V' of $\mathcal{F}_m$ belong to U then $d(V, V') \geq 2^{-j(U)}$.

(b) for every $V \in \mathcal{F}_m$ such that $V \subset U$, there exist a cube $\widetilde{V} \in \mathcal{Q}(U, j(U), \delta(U)) \setminus \widetilde{\mathcal{Q}}(U, j(U), \delta(U))$ such that $V \subset \widetilde{V} \subset U$, as well as an element $x(\widetilde{V}) \in \widetilde{V}$ and a positive real number $r(\widetilde{V})$ satisfying $(x(\widetilde{V}), r(\widetilde{V})) \in \mathcal{S}$ and $2^{-j(U)-1} \leq r(\widetilde{V}) \leq 2^{-j(U)}$. Moreover, every element $x \in V$ is approximated at a rate belonging to $[\delta(U), \delta(U) + \varphi(2^{-j(U)})]$ by $(x(\widetilde{V}), r(\widetilde{V}))$.

(c) if $p \geq 1$ satisfies $g(U) \leq r_p \leq g(V)$ and $x_p \neq x(\widetilde{V})$, then no element $x \in V$ belongs to $B\big(x_p, r_p^{\delta(U)+\theta(2^{-j(U)})H_\varepsilon/2}/16\big)$.

(3) If $1 \leq m \leq n$ and $U \in \mathcal{F}_{m-1}$, then for $V \in \mathcal{F}_m$ such that $V \subset U$ we have:

$$\mu_\varepsilon(V) = \frac{\mu_\varepsilon(U)}{\#\{V' \in \mathcal{F}_m : \ V' \subset U\}}.$$

(4) For all $1 \leq m \leq n$, $U \in \mathcal{F}_{m-1}$ then for $V \in \mathcal{F}_m$ such that $V \subset U$ we have

$$\mu_\varepsilon(V) \leq |V|^{d/\delta(U) - 3d\varphi(|V|)}.$$

Parts 1. to 4. of the induction are easily checked for the first generations $\mathcal{F}_1$ and $\mathcal{F}_2$.

The technique we use to build the generation $\mathcal{F}_{n+1}$ is the same as for the first iteration. We briefly indicate the steps to follow.

For each $U_n \in \mathcal{F}_n$, we know that there are infinitely many integers $j \geq g(U_n)$ such that

$$\begin{aligned} \#\mathcal{Q}(U_n, j, \delta(U_n)) &\geq \kappa \cdot 2^{d(j-g(U_n))} \\ \#\widetilde{\mathcal{Q}}(U_n, j, \delta(U_n)) &\leq C_d \cdot 2^{d(j-g(U_n))}\Big(2^{-dj\varphi(2^{-j})} \\ &\quad + \sum_{g(U_n) \leq k \leq j/\delta(U_n)} 2^{-dk(\delta(U_n)-1-\psi(2^{-k}))}\Big). \end{aligned}$$

If the integer $j = j(U_n)$ is chosen large enough, there is a set $\widetilde{\mathcal{F}}_{n+1}(U_n)$ of cardinality $\frac{\kappa}{2^{d+1}} \cdot 2^{d(j-g(U_n))}$ included in $\mathcal{Q}(U_n, j, \delta(U_n)) \setminus \widetilde{\mathcal{Q}}(U_n, j, \delta(U_n))$ such that

- the dyadic cubes $\widetilde{V}$ belonging to $\widetilde{\mathcal{F}}_{n+1}(U_n)$ are mutually distant from at least $2^{-j(U_n)}$,
- each $\widetilde{V} \in \widetilde{\mathcal{F}}_{n+1}(U_n)$ satisfies simultaneously $\widetilde{V} \in \mathcal{G}_j$, $\widetilde{V} \subset U_n$, $\mathcal{P}\big(\widetilde{V}, \delta(U_n)\big)$ and

$$\widetilde{V} \bigcap \left(\bigcup_{k=g(U_n)}^{\gamma(j)} \bigcup_{p \in \mathcal{T}_k} B(y_p, (\rho_p)^{\delta(U_n)}) \right) = \emptyset.$$

We can associate with each $\widetilde{V} \in \widetilde{\mathcal{F}}_{n+1}(U_n)$ a contracted descendant V, which is a dyadic cube enjoying the properties:

- By condition $\mathcal{C}_2$, there is $x(\widetilde{V}) \in \widetilde{V}$ and a positive real number $r(\widetilde{V})$ satisfying $(x(\widetilde{V}), r(\widetilde{V})) \in \mathcal{S}$ and $2^{-j(U_n)-1} \leq r(\widetilde{V}) \leq 2^{-j(U_n)}$. Moreover, every element $x \in V$ is approximated at a rate belonging to $[\delta(U_n), \delta(U_n) + \varphi(2^{-j(U_n)})]$ by $(x(\widetilde{V}), r(\widetilde{V}))$.

- if $p\geq 1$ is such that $2^{-g(V)}\leq r_p\leq 2^{-g(U_n)}$ and $x_p\neq x(\widetilde{V})$, then for all $x\in V$ we have $x\notin B\big(x_p,(r_p)^{\delta(U_n)+\theta(2^{-j(U_n)})H_\varepsilon/2}/16\big)$.

Then we set

$$\mathcal{F}_{n+1}(U_n)=\{V: V \text{ is the contracted descendant of some } \widetilde{V}\in\widetilde{\mathcal{F}}_{n+1}(U_n)\},$$

and

$$\mathcal{F}_{n+1}=\{V\in\mathcal{G}:\ \exists U_n\in\mathcal{F}_{n+1} \text{ such that } V\in\mathcal{F}_{n+1}(U_n)\}.$$

The measure μ_ε can be extended into a Borel probability measure on the algebra $\sigma_{n+1}=\sigma(L: L\in\bigcup_{p=0}^{n+1}\mathcal{F}_p)$ by the following formula:

$$\text{for every } U\in\mathcal{F}_{n+1}, \text{ for every } V\in\mathcal{F}_{n+1}(U),\quad \mu_\varepsilon(V)=\frac{\mu_\varepsilon(U)}{\#\mathcal{F}_{n+1}(U)}.$$

In addition, requiring that $j_n:=\min(j(U_n): U_n\in\mathcal{F}_n)$ is large enough so that for all $U\in\mathcal{F}_n$ and $T\in\mathcal{F}_{n-1}$ such that $U\subset T$, we obtain that $j(U)\geq 2g(U)$ and

$$\max\left(\frac{2^{d+1}}{\kappa},2^{dg(U)},|U|^{d/\delta(T)-3d\varphi(|U|)}\right)\leq 2^{dj_n\varphi(2^{-j_n})}.$$

This ensures that (30) holds with $p=n+1$. Finally the same lines of computations as in the second step of the construction yield the part 4. of the induction, i.e. the scaling behavior of the measure μ_ε on the dyadic cubes of the $(n+1)$th generation of the Cantor set.

Iterating the previous construction, the Kolmogorov extension theorem yield a measure μ_ε on the algebra $\sigma\Big(V:\ V\in\bigcup_{n\geq 1}\mathcal{F}_n\Big)$ such that all the properties 1. to 4. hold true for all $n\geq 1$. By construction, the measure μ_ε is carried by the Cantor set

$$\mathscr{K}_\varepsilon=\bigcap_{n\geq 0}\bigcup_{V\in\mathcal{F}_n}V.$$

3.6. **Scaling properties of μ_ε.** Let $\delta_\varepsilon=\sup_{U\in\bigcup_{n\geq 0}\mathcal{F}_n}\delta(U)$. By (22), $\delta_\varepsilon\leq H_\varepsilon:=f(x_\varepsilon)+\varepsilon(2\frac{d}{h^2}+1)$.

We are going to show that there exists $C'>0$ such that for every open cube $B\subset[0,1]$,

$$\mu_\varepsilon(B)\leq C'|B|^{d/\delta_\varepsilon}|B|^{-4d\varphi(|B|)}. \tag{31}$$

If (31) holds true, then Lemma 3.3, known as the *mass distribution principle* [17], allows to bound by below the Hausdorff dimension of the support of μ_ε.

Lemma 3.3. *Let F be a Borel set in $\mathbb{R}^d$, and μ be a Borel probability measure on F. Suppose that, for some $\eta>0$, there are $\alpha>0$ and a gauge function ζ such that* $\liminf_{x\to 0^+}\frac{\zeta(x)}{x^\alpha}>0$ *and for every set U with a diameter less than η, $\mu(U)\leq C\zeta(|U|)$.*

Then $\mathcal{H}^\zeta(F)\geq\mu(F)/C$ and $\dim_{\mathcal{H}}F\geq\alpha$.

Let B be an open subcube of $[0,1]^d$ intersecting $\mathscr{K}_\varepsilon$. Let n_0 be the smallest integer such that B intersects at least two elements of $\mathcal{F}_{n_0}$. By construction, the elements V of $\mathcal{F}_{n_0}$ intersecting B are all contained in the same element U of $\mathcal{F}_{n_0-1}$, and $\mu_\varepsilon(B) \leq \mu_\varepsilon(U)$.

Suppose first that $|B| \geq |U|$. Part 4. of the induction yields

$$\mu_\varepsilon(B) \leq \mu_\varepsilon(U) \leq |U|^{d/\delta_\varepsilon - 3d\varphi(|U|)} \leq |B|^{d/\delta_\varepsilon - 3d\varphi(|B|)}$$

when $|B|$ is small enough. Once again the monotonicity of $r \mapsto r^{-\varphi(r)}$ has been used.

Suppose now that $|B| < |U|$. Applying Part 4. of the induction, we find

$$\mu_\varepsilon(B) \leq \mu_\varepsilon(U) \frac{\#\{V \in \mathcal{F}_{n_0} : V \subset U,\ V \cap B \neq \emptyset\}}{\#\{V \in \mathcal{F}_{n_0} :\ V \subset U\}}.$$

Let us use Part 2. of the induction to bound by above $\#\{V \in \mathcal{F}_{n_0} : V \subset U,\ V \cap B \neq \emptyset\}$. There exists an integer $j(U)$ such that the elements of $\mathcal{F}_{n_0}$ that intersect B are distant from one another by at least $2^{-j(U)}$ and have diameter less than $2^{-j(U)}$. Consequently, due to Lemma 2.7.1, there are at most $C_d|B|^d 2^{dj(U)}$ of them.

In addition, we know that

$$\#\{V \in \mathcal{F}_{n_0},\ V \subset U\} \geq \frac{\kappa}{2^{d+1}} \cdot 2^{-dg(U)} 2^{dj(U)} \ = \frac{\kappa}{2^{d+1}} \cdot |U|^d 2^{dj(U)}.$$

This yields thanks to (30)

$$\mu_\varepsilon(B) \ \leq \ \mu_\varepsilon(U) \frac{2|B|^d 2^{dj(U)}}{\frac{\kappa}{2^{d+1}}|U|^d 2^{dj(U)}} \leq 2 \cdot \mu_\varepsilon(U) \frac{|B|^d}{|U|^d} |U|^{-\varphi(|U|)}.$$

Using the scaling behavior of μ_ε on the elements of $\mathcal{F}_{n_0}$, we get

$$\begin{aligned}
\mu_\varepsilon(B) \ &\leq \ 2 \cdot |U|^{d/\delta_\varepsilon - 3d\varphi(|U|)} \frac{|B|^d}{|U|^d} |U|^{-d\varphi(|U|)} \\
&\leq \ 2 \cdot |B|^{d/\delta_\varepsilon - 4d\varphi(|B|)} \Big(\frac{|B|}{|U|}\Big)^{d - d/\delta_\varepsilon} \frac{|B|^{4d\varphi(|B|)}}{|U|^{4d\varphi(|U|)}} \\
&\leq \ 2 \cdot |B|^{d/\delta_\varepsilon - 4d\varphi(|B|)},
\end{aligned}$$

the last line following from the observation that $\Big(\frac{|B|}{|U|}\Big)^{d-d/\delta_\varepsilon} \frac{|B|^{4\varphi(|B|)}}{|U|^{4\varphi(|U|)}}$ is bounded by above by 1 due to the monotonicity property of $r^{\varphi(r)}$ and the fact that $|B| < |U|$.

By the mass distribution principle, the Hausdorff dimension of μ_ε (and thus the Hausdorff dimension of $\mathscr{K}_\varepsilon$) is larger than $\dfrac{d}{\delta_\varepsilon}$, which by (22) is greater than

$$\begin{aligned} \frac{d}{\delta_\varepsilon} &\geq \frac{d}{H_\varepsilon} \geq \frac{\delta}{f(y_\varepsilon)+\varepsilon(2\frac{d}{h^2}+1)} \geq \frac{d}{f(y_\varepsilon)}\cdot \frac{1}{1+\varepsilon(2\frac{d}{h^2}+1)/f(y_\varepsilon)} \\ &\geq (h-\varepsilon)\cdot \frac{1}{1+\varepsilon(2\frac{d}{h^2}+1)/f(y_\varepsilon)} := h_\varepsilon. \end{aligned} \tag{32}$$

It is obvious that h_ε increases toward h when ε goes to zero, hence the result.

3.7. **Relation with $\widetilde{\mathcal{L}}(\Omega\setminus E, f)$.** Let us prove that $\mathscr{K}_\varepsilon \setminus E \subset \widetilde{\mathcal{L}}(\Omega\setminus E, f)$.

Let $x\in \mathscr{K}_\varepsilon\setminus E$ and for $n\geq 1$ denote by $U_n(x)$ the unique element of $\mathcal{F}_n$ that contains x. Using parts 2. and 3. of the induction, we have $\delta_x = \limsup_{n\to\infty}\delta(U_n(x))$.

Recall that the function f is continuous at x. Using formula (20), one observes that $f(y_{U_n(x)})$ converges to $f(x)$ (since $y_{U_n(x)}$ is any point of $U_n(x)$). This implies that $\delta(U_n(x))$ converges to $f(x)$ when n tends to infinity.

Finally, $\delta_x = \lim_{n\to\infty}\delta(U_n(x)) = f(x)$.

3.8. **Density of $\widetilde{\mathcal{L}}(\Omega\setminus E, f)$ in Ω when $\dim_{\mathcal{H}} E < \frac{d}{\sup\{f(x):x\in\Omega\setminus E\}}$.** Using what precedes, we are able to construct a Cantor set $\mathcal{K}_\varepsilon$ in order to approximate the Hausdorff dimension of $\widetilde{\mathcal{L}}(\Omega\setminus E, f)$. But our construction may be achieved in a neighborhood U_y of any point $y\in\Omega\setminus E$ such that $\dim_{\mathcal{H}} E\cap U_y < \frac{d}{\inf\{f(x):x\in U_y\setminus E\}}$. Consequently, if $\dim_{\mathcal{H}} E < \frac{d}{\sup\{f(x):x\in\Omega\setminus E\}}$ then we get the conclusion, since $\Omega\setminus E$ is dense in Ω.

4. Upper bounds for the dimensions

We suppose that the assumptions of Theorem 1.3 are fulfilled. As in the previous section, we set $\delta = \inf\{f(x): x\in\Omega\setminus E\}$ and $h = d/\delta$.

By (6), we know that $\delta_x\geq 1$ for all $x\in\Omega$. The set $\widetilde{\mathcal{L}}(\Omega\setminus E, f)$ contains only elements $x\in[0,1]^d$ satisfying $f(x)\geq\delta$, which implies that $\delta(x)\geq\delta$. Hence, for every $\varepsilon>0$, $\widetilde{\mathcal{L}}(\Omega\setminus E, f)\subset \mathcal{L}_{\delta-\varepsilon}$, where we recall that $\mathcal{L}_{\delta-\varepsilon}$ is given by (2):

$$\mathcal{L}_{\delta-\varepsilon} = \bigcap_{N\geq 1}\bigcup_{n\geq N} B(x_n, r_n^{\delta-\varepsilon}) = \bigcap_{N\geq 1}\bigcup_{n\geq N} B(y_n,\rho_n^{\delta-\varepsilon}).$$

It is known (see [3]) that if the system $\mathcal{S}$ satisfies $\mathcal{C}_1$, then $\dim_{\mathcal{H}}\mathcal{L}_\delta\leq d/\delta$ for all $\delta\geq 1$. Let us prove it briefly for completeness.

Let $s>d/\delta$. For any integer $N\geq 1$, a covering of the limsup set $\mathcal{L}_\delta$ is provided by the union of sets $\bigcup_{n\geq N} B(y_n,\rho_n^\delta)$. Let $\eta>0$, and choose N large enough so that $2\rho_n\leq\eta$ for $n\geq N$. Recalling the definition of the generalized Hausdorff measure associated with the gauge function $\zeta_s(x)=x^s$, we see that

$$\mathcal{H}_\eta^{\zeta_s}(\mathcal{L}_\delta)\leq\sum_{n\geq N}|B(y_n,\rho_n^\delta)|^s\leq\sum_{j=J}^{+\infty}\sum_{p\in\mathcal{T}_j}|B(y_p,\rho_p^\delta)|^s\leq\sum_{j=J}^{+\infty}\sum_{p\in\mathcal{T}_j}2^{-js\delta},$$

where J is the unique integer such that $y_N \in \mathcal{T}_J$. Using Condition $\mathcal{C}_1$, we see that

$$\mathcal{H}^{\zeta_s}_\eta(\mathcal{L}_\delta) \leq \sum_{j=J}^{+\infty} N_j \cdot 2^{dj-js\delta} \leq \sum_{j=J}^{+\infty} N_j \cdot 2^{j(d-s\delta)}.$$

This series converges, since $\log(N_j) = o(j)$ and $d - s\delta < 0$ by construction. Consequently, the s-Hausdorff measure of $\mathcal{L}_\delta$ is finite for any $s > d/\delta$. This demonstrates that $\dim_{\mathcal{H}} \mathcal{L}_\delta \leq d/\delta$.

The above argument applies to $\mathcal{L}_{\delta-\varepsilon}$ when $\delta - \varepsilon > 1$, and thus

$$\dim_{\mathcal{H}} \widetilde{\mathcal{L}}(\Omega \setminus E, f) \leq \inf_{\varepsilon>0} \dim_{\mathcal{H}} \mathcal{L}_{\delta-\varepsilon} \leq \inf_{\varepsilon>0} d/(\delta-\varepsilon) = d/\delta = h.$$

This yields the conclusion.

5. Examples of suitable systems $\big((x_n, r_n)\big)_{n\geq 1}$

5.1. **Approximation by b-adic numbers.** We prove that the dyadic system satisfies $\mathcal{C}_1$ and $\mathcal{C}_2$. The case of the b-adic system (whose definition is clear) is similar.

Define the system $\mathcal{D} = \big((\mathbf{k}\cdot 2^{-j}, 2^{-j})\big)_{j\geq 1, \mathbf{k}\in\{0,1,\dots,2^{-j}-1\}^d}$, and consider the approximation rate of any $x \in [0,1]^d$ by $\mathcal{D}$

$$\delta_x = \sup\{\delta \geq 1 : |x - \mathbf{k}\cdot 2^{-j}| \leq 2^{-j\delta} \text{ for an infinite number of } (j,\mathbf{k})\}.$$

We rather consider the system $\mathcal{D}' = \big((\mathbf{k}\cdot 2^{-j}, \frac{2^{-j}}{32})\big)_{j\geq 1, \mathbf{k}\in\{0,\dots,2^{-j}-1\}^d}$ and the associated approximation rate

$$\delta'_x = \sup\{\delta \geq 1 : |x - \mathbf{k}\cdot 2^{-j}| \leq (\frac{2^{-j}}{32})^\delta \text{ for an infinite number of } (j,\mathbf{k})\}.$$

Of course, $\delta_x = \delta'_x$ for every $x \in [0,1]^d$, but the constant 32 is necessary for our condition $\mathcal{C}_2$ to hold.

The irreducible subsystem of $\mathcal{D}'$ consists in the couples $(\mathbf{k}\cdot 2^{-j}, \frac{2^{-j}}{32})$ for which at least one coordinate of $\mathbf{k}$ is odd. Therefore, it is obvious that the weak redundancy condition $\mathcal{C}_1$ is satisfied, the corresponding sequence $(N_j)_{j\geq 1}$ being constant equal to 1, so that $\gamma(j) = j$ for every $j \geq 1$.

To check $\mathcal{C}_2$, let $\delta > 1$, and consider any $\varphi \in \Phi$. Let $\mathbf{k}\cdot 2^{-j}$ be a dyadic element of $[0,1]^d$ such that $\mathbf{k}$ has at least one odd coordinate. We call V the dyadic cube $\prod_{i=1}^d [k_i \cdot 2^{-j}, (k_i+1)\cdot 2^{-j})$ of generation $g(V) = j$. Given a dyadic generation j, the number of such dyadic irreducible cubes is greater than 2^{dj-1}. Then the property $\mathcal{P}(V,\delta)$ holds without any further condition. Indeed, we only have to check that for every $j' \in \{\gamma(j)j, ..., (j+1)\delta + 4\}$, for every $\mathbf{k}'$ (with at least one odd coordinate), $\mathbf{k}' 2^{-j'} \notin B(\mathbf{k}\cdot 2^{-j}, (\frac{2^{-j}}{32})^\delta)$.

This is obvious, since by the structure of the dyadic tree we get when $j \leq j' \leq (j+1)\delta + 4$

$$|\mathbf{k} \cdot 2^{-j} - \mathbf{k}' \cdot 2^{-j'}| \quad \geq \quad 2^{-j'} \geq 2^{-(j+1)\delta - 4} \geq \frac{2^{-j\delta}}{16 \cdot 2^{\delta}} \geq (\frac{2^{-j}}{32})^{\delta},$$

and when $\gamma(j) \leq j' < j$

$$|\mathbf{k} \cdot 2^{-j} - \mathbf{k}' \cdot 2^{-j'}| \quad \geq \quad 2^{-j} \geq (\frac{2^{-j}}{32})^{\delta}.$$

Thus the system $\mathcal{D}$ satisfies $\mathcal{C}_2$ with a function κ constant equal to $1/2$ (it holds for all the "irreducible" sub-cubes of $[0,1]^d$).

5.2. **Diophantine approximation by rational numbers in $\mathbb{R}$.** Consider the system

$$\mathcal{R} = \big((p/q, 1/q^2)\big)_{q\geq 1,\, 0\leq p\leq q-1}.$$

It follows from Dirichlet's argument that $\mathcal{L}_1(\mathcal{R}) = [0,1]$. The irreducible sub-system of $\mathcal{R}$ consists in the elements of $\mathcal{R}$ such that $p \wedge q = 1$.

We are going to check that $\mathcal{R}$ satisfies $\mathcal{C}_1$ and $\mathcal{C}_2$.

Let $j \geq 1$ be an integer, and let $(p/q, 1/q^2) \in \mathcal{R}$ be such that $q^2 \in (2^j, 2^{j+1}]$. We shall prove that $B(p/q, 1/q^2)$ may contain only a bounded number of rational numbers p'/q' satisfying $(p'/q', 1/(q')^2) \in \mathcal{R}$ and $(q')^2 \in (2^j, 2^{j+1}]$. This implies $\mathcal{C}_1$.

If $p/q \neq p'/q'$, then one has necessarily that $|p/q - p'/q'| = |pq' - p'q|/(qq') \geq 1/(qq') \geq 2^{-j-1}$, since q and q' belong to $[2^{j/2}, 2^{(j+1)/2})$. Since the diameter of $B(p/q, 1/q^2)$ is at most 2^{-j+1}, there are at most 4 distinct irreducible rational numbers p'/q' belonging to $B(p/q, 1/q^2)$. Hence $\mathcal{R}$ satisfies $\mathcal{C}_1$, with a sequence $(N_j)_{j\geq 1}$ constant equal to 4.

In order to prove $\mathcal{C}_2$, we consider $V := [K \cdot 2^{-J}, (K+1) \cdot 2^{-J})$ a dyadic interval in $[0,1]$ of generation J, a real number $\delta > 1$ and any function $\varphi \in \Phi$. We demonstrate that $\mathcal{P}(V, \delta)$ holds without any restriction on V, δ and φ. Obviously V contains a rational number p/q satisfying $q^2 \in (2^J, 2^{J+1}]$ (p/q is not necessarily irreducible). Assume that a rational number $p'/q' \neq p/q$ belongs to $B(p/q, 1/q^{2\delta})$ with $\log_2((q')^2) \in [\gamma(J), \cdots, \delta(J+1)+4]$. This implies that $q' \leq 2^{(\delta(J+1)+4)/2} \leq q^{\delta/2} 2^{\delta/2+2}$. Combining the information, we have

$$1/q^{2\delta} \geq |p/q - p'/q'| \geq 1/(qq') \geq 2^{-\delta/2-2}/q^{1+\delta/2}.$$

This last inequalities can not hold as soon as $\delta > 1$ (provided that q is large enough). Consequently, $\mathcal{P}(V, \delta)$ holds, and $\mathcal{R}$ satisfies $\mathcal{C}_2$ with a function κ constant equal to 1.

5.3. **Inhomogeneous Diophantine approximation.** Let α be an irrational number in $[0,1]$. Consider the system

$$\mathcal{I} = \Big((\{n\alpha\}, \frac{1}{n}) \Big)_{n\geq 1},$$

where $\{x\}$ stands for the fractional part of the real number x.

It is proved in [3] (Proposition 6.1) that $\mathcal{I}$ satisfies $\mathcal{C}_1$ if and only if the approximation rate of α by the rational system $\mathcal{R}$ equals 2.

We prove that $\mathcal{I}$ satisfies $\mathcal{C}_2$, when the approximation rate of α by the rational system $\mathcal{R}$ is 2. When this holds, for every $\varepsilon > 0$, there is an integer q_ε such that

$$\text{(33)} \qquad \text{for every } q \geq q_\varepsilon, \text{ for every integer } p, \ |\alpha - p/q| \geq 1/q^{1+\varepsilon}.$$

We focus now on $\mathcal{C}_2$. For this, let us recall the *three distance theorem* [37, 36, 14]: the real numbers $\{\alpha\}$, $\{2\alpha\}$, $\{3\alpha\}$, ..., $\{N\alpha\}$ divide the interval $[0,1]$ into $N+1$ intervals whose lengths take at most three values $d_1(N)$, $d_2(N)$ and $d_3(N)$, satisfying

$$d_1(N) < d_2(N) < d_3(N) \leq \frac{3}{N+1}.$$

Let $J \geq 1$. As for the rational system, in order to prove $\mathcal{C}_2$, we consider $V := [K \cdot 2^{-J}, (K+1)\cdot 2^{-J})$ a dyadic interval in $[0,1]$ of generation J, a real number $\delta > 1$ and any function $\varphi \in \Phi$. We demonstrate that $\mathcal{P}(V,\delta)$ holds without any restriction on V, δ and φ for a sufficiently large number of dyadic intervals V.

Apply the three distance theorem to $\{\alpha\}$, $\{2\alpha\}$, $\{3\alpha\}$, ..., $\{2^J\alpha\}$. The 2^J+1 corresponding intervals of $[0,1]$ have length less than $3/(2^J+1)$. By a translation argument, the points $\{(2^J+1)\alpha\}$, $\{(2^J+2)\alpha\}$, $\{(2^J+3)\alpha\}$, ..., $\{2^{J+1}\alpha\}$ divide the interval $[0,1]$ into 2^J+1 intervals whose lengths are also less than $3/(2^J+1)$. This means that among the dyadic intervals of generation J, there are no three consecutive dyadic intervals U which do not contain one of the points $\{n\alpha\}$, for n ranging over $\{2^J+1, 2^J+2, ..., 2^{J+1}\}$.

Let us consider one such interval $V := [K\cdot 2^{-J}, (K+1)\cdot 2^{-J})$, which contains $\{n\alpha\}$ for some n belonging to $\{2^J+1, 2^J+2, ..., 2^{J+1}\}$. Assume that another point $\{n'\alpha\}$ belongs to $B(\{n\alpha\}, 1/n^\delta)$ with $\log_2 n' \in [\gamma(J), \cdots, [\delta(J+1)]+4]$. This means that $|\{n'\alpha\} - \{n\alpha\}| \leq \frac{1}{n^\delta}$. By definition there are integers p and p' satisfying $n\alpha = p + \{n'\alpha\}$ and $n'\alpha = p' + \{n\alpha\}$, hence

$$|(n'\alpha + p') - (n\alpha - p)| = |(n-n')\alpha - (p'-p)| \leq \frac{1}{n^\delta},$$

or equivalently

$$\Big|\alpha - \frac{p'-p}{n'-n}\Big| \leq \frac{1}{|n'-n|\cdot n^\delta} \leq \frac{C}{|n'-n|^{\delta+1}},$$

the last inequality following from the fact that $|n'-n| \leq 2\delta \cdot n$. This contradicts (33). Consequently, $\mathcal{P}(V,\delta)$ holds.

Finally, $\mathcal{I}$ satisfies $\mathcal{C}_2$ with a function κ constant equal to 1/3.

5.4. **Poisson point process.** Let $\mathcal{P}$ be a Poisson point process with intensity

$$\Lambda = \mathbf{1}_{[0,1]\times(0,1)}(x,y)\cdot \ell(dx)\otimes \frac{\ell(dy)}{y^2}, \tag{34}$$

where ℓ stands for the Lebesgue measure on $(0,1)$. We rewrite $\mathcal{P}$ as $\mathcal{P} = \big((x_n,r_n)\big)_{n\geq 1}$, where $(r_n)_{n\geq 1}$ is a positive decreasing sequence converging to zero when n tends to infinity.

With probability one, such a system satisfies $\mathcal{C}_1$, see for instance Proposition 6.2 in [3].

We now deal with $\mathcal{C}_2$. We only need to find a function $\varphi\in\Phi$ and a continuous function $\kappa:(1,+\infty)\to\mathbb{R}_+^*$ such that for every $\delta>1$, with probability 1, for every $U\in\mathcal{G}_*$, there are infinitely many integers $j\geq g(U)$ satisfying $\#\mathcal{Q}(U,j,\delta)\geq\kappa(\delta)\cdot 2^{j-g(U)}$. Then, for any countable and dense subset Δ of $(1,\infty)$, with probability 1, for every $\delta\in\Delta$, for every $U\in\mathcal{G}_*$, there are infinitely many integers $j\geq g(U)$ satisfying $\#\mathcal{Q}(U,j,\delta)\geq\kappa(\delta)\cdot 2^{j-g(U)}$.

In fact, any $\varphi\in\Phi$ is suitable.

Let $\varphi\in\Phi$ and $\delta>1$. For $U\in\mathcal{G}_*$ and $V\subset U$ such that $V\in\bigcup_{j>g(U)}\mathcal{G}_j$, let us introduce the event

$$\mathcal{A}(U,V,\delta)=\left\{\begin{matrix}\exists\; n\in\mathcal{T}_{g(V)} \text{ such that } \; x_n\in V \text{ and} \\ B(x_n,(r_n)^\delta)\bigcap\left(\bigcup_{\gamma(g(V))\leq k\leq h(V)}\mathcal{T}_j\right)=\{x_n\}\end{matrix}\right\}$$

where $h(V)=\big[\delta(g(V)+1)\big]+4$. Recall that $n\in\mathcal{T}_{g(V)}$ means that $2^{-g(V)-1}<r_n\leq 2^{-g(V)}$. Note that by construction, we have the inclusion $\mathcal{A}(U,V,\delta)\subset\{\mathcal{P}(V,\delta)\text{ holds}\}$.

For every $j\geq 1$, let $\widetilde{\mathcal{G}}_j=\big\{[2k\cdot 2^{-j},(2k+1)\cdot 2^{-j}]:0\leq k\leq 2^j-1\big\}$. The restrictions of the Poisson point process to the strips $V\times(0,1)$, where V describes $\widetilde{\mathcal{G}}_j$, are independent. Consequently, the events $\mathcal{A}(U,V,\delta)$, when $V\in\widetilde{\mathcal{G}}_j$ and $V\subset U$, are independent (we must separate the intervals in $\widetilde{\mathcal{G}}_j$ because if $V\in\mathcal{G}_j$, $x_n\in V$ and $r_n\leq 2^{-j}$, then $B(x_n,(r_n)^\delta)$ may overlap with the neighbors of V).

We denote by $X(U,V,\delta)$ the random variable $\mathbf{1}_{\mathcal{A}(U,V,\delta)}$. For a given generation $j>g(U)$, the random variables $(X(U,V,\delta))_{V\in\widetilde{\mathcal{G}}_j}$ are i.i.d Bernoulli variables, whose common parameter is denoted by $p_j(\delta)$. We have the following Lemma.

Lemma 5.1. *There exists a continuous function $\kappa_1:(1,+\infty)\to(0,1)$ such that for every $j\geq 1$, $p_j(\delta)\geq\kappa_1(\delta)$.*

Let us assume Lemma 5.1 for the moment. By definition we have

$$\#\mathcal{Q}(U,j,\delta)\geq\sum_{V\in\widetilde{\mathcal{G}}_j:\;V\subset U}X(U,V,\delta).$$

The right hand side in the last inequality is a binomial variable of parameters $(2^{j-g(U)}, p_j(\delta))$, with $p_j(\delta) \geq \kappa_1(\delta) > 0$. Consequently, there exists a constant $\kappa(\delta) > 0$ satisfying

$$\mathbb{P}\Big(\sum_{V \in \mathcal{G}_j,\ V \subset U} X(U,V,\delta) \geq \kappa(\delta) \cdot 2^{j-g(U)} \Big) \geq 1/2 \tag{35}$$

provided that j large enough. The continuity of κ with respect to the parameter $\delta > 1$ follows from the continuity of κ_1.

Let $(j_n)_{n\geq 1}$ be the sequence defined inductively by $j_1 = g(U) + 1$ and $j_{n+1} = (j_n + 1)\delta + 5$. We notice that the events E_n defined for $n \geq 1$ by

$$E_n = \{\#\mathcal{Q}(U, j_n, \delta) \geq \kappa(\delta) \cdot 2^{j_n - g(U)}\}$$

are independent. Moreover, (35) implies that $\sum_{n\geq 1} \mathbb{P}(E_n) = +\infty$. The Borel-Cantelli Lemma yields that, with probability 1, there is an infinite number of generations j_n satisfying $\#\mathcal{Q}(U, j_n, \delta) \geq \kappa(\delta) \cdot 2^{j-g(U)}$. This holds true for every $U \in \mathcal{G}_*$ almost surely, hence almost surely for every $U \in \mathcal{G}_*$. Condition $\mathcal{C}_2$ is proved.

We prove Lemma 5.1. For every $V \in \mathcal{G}_*$, let us introduce the sets

$$S_V = V \times [2^{-(g(V)+1)}, 2^{-g(V)}] \ \text{ and } \ \widetilde{S}_V = V \times [2^{-h(V)}, 2^{-\gamma(g(V))}].$$

We denote by N_V and $\widetilde{N}_V$ respectively the cardinality of $\mathcal{P} \cap S_V$ and $\mathcal{P} \cap (\widetilde{S}_V \setminus S_V)$. These random variables N_V and $\widetilde{N}_V$ are independent, and we set $l_V = \Lambda(S_V)$ and $\widetilde{l}_V = \Lambda(\widetilde{S}_V)$ (Λ is the intensity of the Poisson point process (34)). Due to the form of the intensity Λ, N_V and $\widetilde{N}_V$ are Poisson random variables of parameter $l_V = 1$ and $\widetilde{l}_V = 2^{-g(V)}\big(2^{h(V)} - 2^{g(V)+1} + 2^{g(V)} - 2^{\gamma(g(V))}\big)$ respectively. Observe that $\widetilde{l}_V \leq 2^{h(V)-g(V)}$ since by definition $\gamma(g(V)) \leq g(V)$.

We also consider two sequences of random variables in $\mathbb{R}^2$ $(\xi_p = (X_p, Y_p))_{p\geq 1}$ and $(\widetilde{\xi}_q = (\widetilde{X}_q, \widetilde{Y}_q))_{q\geq 1}$ such that

$$\begin{aligned} \mathcal{P} \cap S_V &= \{\xi_p :\ 1 \leq p \leq N_V\} \\ \mathcal{P} \cap (\widetilde{S}_V \setminus S_V) &= \{\widetilde{\xi}_q :\ 1 \leq q \leq \widetilde{N}_V\}. \end{aligned}$$

The event $\mathcal{A}(U,V,\delta)$ contains the event $\widetilde{\mathcal{A}}(U,V,\delta)$ defined as

$$\Big\{ N_V = 1 \text{ and } B\big(X_1, Y_1^\delta\big) \bigcap \{\widetilde{X}_q : 1 \leq q \leq \widetilde{N}_V\} = \{X_1\} \Big\},$$

where $\xi_1 = (X_1, Y_1)$. The difference between $\mathcal{A}(U,V,\delta)$ and $\widetilde{\mathcal{A}}(U,V,\delta)$ is that the latter one imposes that there is one and only one Poisson point in

S_V. We have

$$\begin{aligned}
&\mathbb{P}(\widetilde{\mathcal{A}}(U,V,\delta))\\
= \ & \mathbb{P}\Big(\Big\{B\big(X_1,Y_1^\delta\big)\bigcap\{\widetilde{X}_q : 1\le q\le \widetilde{N}_V\}=\emptyset\,\Big|\,\{N_V=1\}\Big\}\Big)\\
& \times\, \mathbb{P}(\{N_V=1\})\\
= \ & \mathbb{P}\Big(\Big\{\forall\, 1\le q\le \widetilde{N}_V,\ \widetilde{X}_q\notin B\big(X_1,Y_1^\delta\big)\}\ \Big|\ \{N_V=1\}\Big\}\Big)\times e^{-1}.
\end{aligned}$$

where $\mathbb{P}(\{N_V=1\})=e^{-1}$ since N_V is a Poisson random variable of parameter 1. The random variables $\widetilde{X}_q$ are i.i.d. uniformly distributed in V. Thus,

$$\begin{aligned}
&\mathbb{P}\Big(\Big\{\forall\, 1\le q\le \widetilde{N}_V,\ \widetilde{X}_q\notin B\big(X_1,Y_1^\delta\big)\}\ \Big|\ \{N_V=1\}\Big\}\Big)\\
&\ge \mathbb{E}\Big(\Big[1-\frac{\ell\big(B(X_1,Y_1^\delta)\big)}{2^{-g(V)}}\Big]^{\widetilde{N}_V}\Big).
\end{aligned}$$

Observe that, since $\delta>1$, provided that $g(V)$ is large enough, conditionally on $\{N_V\ge 1\}$, $\ell\big(B(X_1,Y_1^\delta)\big)\le 2^{-g(V)\delta}$. This implies that

$$\mathbb{P}(\widetilde{\mathcal{A}}(U,V,\delta))\ge e^{-1}\times\mathbb{E}\Big(\big[1-2^{-g(V)(\delta-1))}\big]^{\widetilde{N}_V}\Big). \tag{36}$$

Let us define $\eta_{g(V)}=2^{-g(V)(\delta-1)}$. Using that $\widetilde{N}_V$ is a Poisson random variable of parameter $\widetilde{l}_V$, a classical calculus shows that (36) can be rewritten as

$$\mathbb{P}(\widetilde{\mathcal{A}}(U,V,\delta))\ge e^{-1}e^{-\widetilde{l}_V\cdot\eta_{g(V)}}.$$

In order to conclude, it suffices to bound from above the product $\widetilde{l}_V\cdot\eta_{g(V)}$. This is achieved by recalling the definition of $h(V)=\big[(g(V)+1)\delta\big]+4$, which implies that

$$\widetilde{l}_V\cdot\eta_{g(V)}\quad\le\quad 2^{h(V)-g(V))}2^{-g(V)(\delta-1)}\le 16\cdot 2^\delta.$$

Thus, $\widetilde{l}_V\eta_{g(V)}$ is bounded from above independently of V by a continuous function of δ. As a conclusion, $\mathbb{P}(\widetilde{\mathcal{A}}(U,V,\delta))$, and thus $\mathbb{P}(\mathcal{A}(U,V,\delta))$, is bounded from below by some quantity $\kappa_1(\delta)$ which is strictly positive and continuously dependent on $\delta>1$. Lemma 5.1 is proved.

5.5. **Diophantine approximation by rational elements in $\mathbb{R}^d$.** The question of the validity of $\mathcal{C}_1$ and $\mathcal{C}_2$ may be asked for the rational system in higher dimension $[0,1]^d$

$$\mathcal{R}^d=\big((p_1/q,\dots,p_d/q),1/q^{1+1/d})\big)_{q\ge 1,0\le p_i\le q-1}.$$

Again, it follows from Dirichlet's argument that $\mathcal{L}_1(\mathcal{R}^d)=[0,1]^d$ (see for instance Theorem 200 in [21]). The irreducible sub-system of $\mathcal{R}$ consists in the elements of $\mathcal{R}$ such that $p_i\wedge q=1$ for some i.

It is known that $\dim(\mathcal{L}_\delta(\mathcal{R}^d))=d/\delta$, see [29, 16, 11]. Using this result, the upper bounds in Theorems 1.2 and 1.3 can be proved. Unfortunately we could not demonstrate neither the weak redundancy property nor $\mathcal{C}_2$ for $\mathcal{R}^d$ (or for any reasonable sub-systems of $\mathcal{R}^d$), so we could not obtain the lower bound.

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Julien Barral, Projet SISYPHE, INRIA Rocquencourt, B.P. 105, 78153 Le Chesnay Cedex, France - Stéphane Seuret, LAMA, CNRS UMR 8050, Université Paris-Est, 61 avenue du Général de Gaulle, 94 010 CRÉTEIL Cedex, France