THE MULTIFRACTAL NATURE OF HETEROGENEOUS SUMS OF DIRAC MASSES

Abstract. This article deals with the natural problem of performing the multifractal analysis of heterogeneous sums of Dirac masses

$$\nu = \sum_{n\geq 0} w_n \, \delta_{x_n},$$

where $(x_n)_{n\geq 0}$ is a sequence of points in $[0,1]^d$ and $(w_n)_{n\geq 0}$ is a positive sequence of weights such that $\sum_{n\geq 0} w_n < \infty$. This problem is solved in the case where the points x_n are roughly uniformly distributed in $[0,1]^d$, and the weights w_n depend on a random self-similar measure μ, a parameter $\rho \in (0,1]$, and a sequence of positive radii $(\lambda_n)_{n\geq 1}$ converging to 0 in the following way

$$w_n = \lambda_n^{d(1-\rho)} \mu\big(B(x_n, \lambda_n^\rho)\big) |\log \lambda_n|^{-2}.$$

The measure ν has a rich multiscale structure. The computation of its multifractal spectrum is related to heterogeneous ubiquity properties of the system $\{(x_n,\lambda_n)\}_n$ with respect to μ.

1. Introduction and motivations

A large literature has been dedicated to the multifractal analysis of continuous singular measures possessing scaling invariance properties (see [13, 30, 32, 29, 7] and references therein), while only a few is known about the multifractal nature of another very natural class of singular measures: the infinite sums of Dirac masses.

In this paper the multifractal analysis of a large class of infinite sums of Dirac masses is performed. These Dirac masses are located at roughly uniformly distributed points in a compact subset of $\mathbb{R}^d$, and they are weighted by using a statistically self-similar multifractal measure. The study of these measures is closely related to the new results on heterogeneous ubiquity established in [11].

Let us start by describing a class of "homogeneous" sums of Dirac masses whose multifractal analysis is worked out in [16].

Let Ω be a compact subset of $\mathbb{R}^d$ such that $\dim \Omega > 0$ (dim stands for the Hausdorff dimension) and $\|.\|$ a norm on $\mathbb{R}^d$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\Omega^{\mathbb{N}}$ and $(\lambda_n)_{n\in\mathbb{N}}$ a non-increasing sequence of positive real numbers converging to 0. For $x \in \Omega$, the approximation degree of x by the family $(x_n)_n$ relatively to $(\lambda_n)_n$ is [16]

$$\deg(x) = \infty \text{ if } x \in \{x_n\} \quad \text{and} \quad \deg(x) = \limsup_{n\to\infty} \frac{\log \|x - x_n\|}{\log \lambda_n} \text{ if } x \notin \{x_n\}.$$

The level sets of the function $\deg(\cdot)$ are then defined by $F_\xi = \{x : \deg(x) = \xi\}$ for $\xi \in (0,+\infty)$. A natural assumption is that $\inf_{x\in\Omega} \deg(x) > 0$, which can be easily normalized to have $\deg(x) \geq 1$ for all $x \in \Omega$. This is equivalent to saying that $\Omega = \bigcup_{\xi\geq 1} F_\xi$, which arises under the assumption

$$\limsup_{n\to\infty} B(x_n, \lambda_n) = \bigcap_{N\geq 0} \bigcup_{n\geq N} B(x_n, \lambda_n) = \Omega, \tag{1}$$

One now considers for $\alpha \geq \dim \Omega$ the measure ν defined by

$$\nu = \sum_{n\geq 0} w_n \, \delta_{x_n} \text{ with } w_n = \begin{cases} |\log \lambda_n|^{-2} \lambda_n^\alpha & \text{if } \alpha = \dim \Omega \\ \lambda_n^\alpha & \text{otherwise} \end{cases}, \tag{2}$$

where δ_x stands for the Dirac mass located at $x \in \mathbb{R}^d$.

Suppose that the system $\{(x_n, \lambda_n)\}_{n\in\mathbb{N}}$ is "sparse" in the sense of [16], i.e.

$$\exists\, C' > 0, \forall\, x \in \Omega, \forall\, j \in \mathbb{N}, \#\{n : 2^{-j} \leq \lambda_n < 2^{-j+1}, x_n \in B(x, 2^{-j})\} \leq C'. \tag{3}$$

Suppose also that Ω can be endowed with a monofractal finite Borel measure m in the sense that there exists $r_0 > 0$ and $C > 0$ such that $\forall x \in \Omega$ and $\forall\, 0 < r \leq r_0$, one has $C^{-1} r^{\dim \Omega} \leq m(B(x, r)) \leq C r^{\dim \Omega}$.

Then the measure ν is finite and its Hausdorff multifractal spectrum is found in [16]. Let us recall the definition of this spectrum. If μ is a positive Borel measure on Ω then the lower Hölder exponent of μ at x is

$$h_\mu(x) = \liminf_{r\to 0^+} \frac{\log \mu(B(x,r))}{\log r}, \tag{4}$$

and the level sets of the lower Hölder exponent function are defined as

$$E_h^\mu = \{x \in \Omega : h_\mu(x) = h\} \quad (h \geq 0).$$

The Hausdorff multifractal spectrum of μ is the mapping $d_\mu : h \geq 0 \mapsto \dim\, E_\mu^h$.

For the measure ν defined above on Ω endowed with the monofractal measure m, if both (1) and (3) hold true, then it is shown in [16] that the lower Hölder exponent of ν is directly deduced from $\deg(x)$ by the relation $h_\nu(x) = \alpha(\deg(x))^{-1}$ (actually, only in the case $\alpha = \dim\,\Omega$ is treated, but the case $\alpha > \dim\,\Omega$ is similar). Then the Hausdorff dimension of the sets E_ν^h is intimately related to the dimension of the sets F_ξ. The main point is that the Hausdorff dimensions of F_ξ can be computed in the context where Ω can be endowed by a monofractal measure m, using results on so-called ubiquitous systems [14]. One obtains the following theorem ([16], Corollary 5):

Theorem 1.1.

(1) *For every $x \in \Omega$, $h_\nu(x) = \alpha(\deg(x))^{-1}$. Equivalently, for every $\xi \in [1, \infty]$, $E_{\alpha/\xi}^\nu = F_\xi$.*

(2) *For every $h \in [0, \alpha]$, $d_\nu(h) = (\dim \Omega)\, h/\alpha$.*

A similar result holds for Lévy subordinator [12, 20]. Indeed, the derivative of such a subordinator takes a form comparable to (2) when restricted to any non-trivial compact subinterval of $\mathbb{R}_+$. Comparable multifractal properties are also obtained for sums of Dirac masses located on dyadic points of $[0, 1]$ in [2, 8], and for functions with countable and dense set of jump points like the Riemann function and general Lévy processes [19, 20].

A central role is played in Theorem 1.1 by the fact that for each n, the weight w_n of the Dirac mass at x_n does not depend on the location (in space) of x_n but on λ_n only. Consequently, due to (1) and (3), the weights w_n are roughly homogeneously distributed. This raises the much more general problem of performing the multifractal analysis of sums of Dirac masses heterogeneously weighted

$$\nu = \sum_{n\in\mathbb{N}} w(x_n, \lambda_n)\, \delta_{x_n},$$

as well as finding the counterpart of the ubiquity properties used in the analysis above to this heterogeneous case.

In this paper, we resolve this problem when the heterogeneity in the weight's distribution is governed by a (possibly) multifractal measure. The set Ω is $[0,1]^d$. We consider a positive and finite Borel measure μ such that $\text{supp}(\mu)=\Omega$ and a parameter $\rho\in(0,1]$. A property slightly weaker than (3) is assumed:

Definition 1.2. *For every $j\geq 0$, let us define*

$$T_j=\left\{n:\ 2^{-(j+1)}<\lambda_n\leq 2^{-j}\right\}. \tag{5}$$

The system $\{(x_n,\lambda_n)\}_{n\in\mathbb{N}}$ is said to be weakly redundant *when there exists a sequence of integers $(N_j)_{j\geq 0}$ such that*

1. $\lim_{j\to\infty}\frac{\log_2 N_j}{j}=0$.
2. *for every $j\geq 1$, T_j can be decomposed into N_j pairwise disjoint subsets (denoted $T_{j,1},\ldots,\ T_{j,N_j}$) such that for each $1\leq i\leq N_j$, the family $\left\{B(x_n,\lambda_n):\ n\in T_{j,i}\right\}$ is composed of disjoint balls. (By convention, $N_j=1$ if $T_j=\emptyset$.)*

When the system $\{(x_n,\lambda_n)\}_{n\in\mathbb{N}}$ is weakly redundant, we shall study the multifractal nature of the finite Borel measure ν_ρ $(0<\rho\leq 1)$ defined as

$$\nu_\rho=\sum_{n\in\mathbb{N}} a_n\,\lambda_n^{d(1-\rho)}\mu\big(B(x_n,\lambda_n^\rho)\big)\,\delta_{x_n}, \tag{6}$$

where $a_n=|\log\lambda_n|^{-2}c_n^{-1}$ and $c_n=N_j$ if $n\in T_j$. The term a_n is a natural normalization factor that satisfies $|\log a_n|=o(|\log\lambda_n|)$ and makes the measure ν_ρ finite.

It is easily observed that up to a multiplicative constant, if μ is the d-dimensional Lebesgue measure ℓ, then for every $\rho\in(0,1]$ the measure ν_ρ coincides with the measure considered in [16], i.e. the measure (2) when $\Omega=[0,1]^d$ and $\alpha=d$. Formula (6) is thus a natural "heterogeneous" extension of the previous "homogeneously distributed" measures. We shall see that an important role is played in general by the dilation parameter ρ: the multifractal behavior of ν_ρ when $\rho<1$ strongly differs from the behavior of ν_1. Let us also mention that a preliminary result is obtained in [8] in the special case when $\rho=1$, $d=1$, and the system $\{(x_n,\lambda_n)\}_n$ is equal to $\{(kb^{-j},2^{-j})\}_{j\geq 0,k\in\{0,\ldots,2^j-1\}}$. There, the hierarchical structure of the dyadic numbers considerably simplifies the discussion with respect to the much more general situation considered in this paper.

In order to get a foretaste of our main result Theorem 2.9, let us expose one of its corollaries. Let μ_0 be a binomial measure with weights $p_0>0$, $p_1>0$, $p_0+p_1=1$. This simple measure μ_0 fulfills the assumptions of Theorem 2.9.

Theorem 1.3. *Let $\Omega=[0,1]$. Let $\{(x_n,\lambda_n)\}_n$ be a weakly redundant system satisfying*

$$\Omega\subset\limsup_{n\to+\infty}B(x_n,\lambda_n/2). \tag{7}$$

Consider the measure ν_1 (6) constructed using the system $\{(x_n,\lambda_n)\}_n$ and μ_0.

Let $\tau(q)=-\log_2(p_0^q+p_1^q)$ for $q\in\mathbb{R}$ and $\tau^*(h)=\inf_{q\in\mathbb{R}}(hq-\tau(q))$ for $h\geq 0$.

1. *If $0\leq h\leq\tau'(1)$, then $d_{\nu_1}(h)=h$.*
2. *Let $h\geq\tau'(1)$. If $\tau^*(h)>0$, then $d_{\nu_1}(h)=d_{\mu_0}(h)=\tau^*(h)$, and if $\tau^*(h)<0$ then $E_h^{\nu_1}=\emptyset$.*

Remark that (7) requires that Ω is covered by the balls of radii $\lambda_n/2$ instead of λ_n in (1). This slight modification is purely technical and comes from the replacement of the Lebesgue measure by μ_0 in (6) with respect to the situation described by Theorem 1.1.

This rather simple case of a binomial measure illustrates a phenomenon that occurs all along the paper. The local regularity of ν_ρ at each point x is ruled simultaneously by the behavior of the measure μ around x and by the approximation degree $\deg(x)$ (by opposition to what happens for the measure ν considered in (2), for which only $\deg(x)$ was determinant). This combination has a repercussion on the shape of d_{ν_1}. In particular, the linear part in the multifractal spectrum is due to a subtle combination studied in [11] between ubiquity properties like (1) and the monodimensionality and self-similarity properties of the measure μ_0.

Before making other comments, our main result Theorem 2.9 must be precisely stated. This requires some definitions and technical conditions explained in Sections 2.1 to 2.3. These conditions are satisfied by large classes of measures μ possessing some statistical self-similarity and by many systems $\{(x_n,\lambda_n)\}_n$ – see Section 5 for more details –.

Theorem 2.9 is then given in Section 2.4 and proved in Sections 3 and 4. Theorem 2.9 extends Theorem 1.1, and asserts that the multifractal spectrum d_{ν_ρ} is composed of a linear part and a concave part, under suitable assumptions on μ and on the system $\{(x_n,\lambda_n)\}_n$. Consequently, the same phenomenon as the one described for ν_1 by Theorem 1.3 happens for more general classes of measures ν_ρ.

2. Statement of the main result

In the sequel, $d \geq 1$ is fixed and $\Omega = [0,1]^d$. One also considers a positive Borel measure μ with $\text{supp}(\mu) = \Omega$, a sequence $(x_n)_n \in \Omega^{\mathbb{N}}$, a non-increasing sequence $(\lambda_n)_n$ of positive real numbers converging to zero and a parameter $0 < \rho \leq 1$.

It is convenient to endow $\mathbb{R}^d$ with the supremum norm $\|\cdot\|_\infty$ and with the associated distance $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d \mapsto \|x-y\|_\infty = \max_{1\leq i\leq d}(|x_i - y_i|)$.

2.1. **Some notations and definitions.** Let c be an integer ≥ 2. For every $j \geq 0$, for every $\mathbf{k} = (k_1,\ldots,k_d) \in \{0,1,\ldots,c^j-1\}^d$, $I^c_{j,\mathbf{k}}$ denotes the c-adic box $[k_1c^{-j},(k_1+1)c^{-j}) \times \ldots \times [k_dc^{-j},(k_d+1)c^{-j})$. For every $x \in [0,1)^d$, $I^c_j(x)$ stands for the unique c-adic box of scale j that contains x, and $\mathbf{k}^c_{j,x}$ is the unique (multi-)integer such that $I^c_j(x) = I^c_{j,\mathbf{k}^c_{j,x}}$. If $\mathbf{k} = (k_1,\ldots,k_d)$ and $\mathbf{k}' = (k'_1,\ldots,k'_d)$ both belong to $\mathbb{N}^d$, $\|\mathbf{k}-\mathbf{k}'\|_\infty = \max_i |k_i - k'_i|$.

All along the paper, if $E \in \mathbb{R}^d$, $|E|$ denotes the diameter of the set E, and if $x \in \mathbb{R}$, $[x]$ stands for the integer part of x.

Let m be a positive Borel measure on $\mathbb{R}^d$ such that $\text{supp}(m) = \Omega$. Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing continuous function such that $\psi(0) = 0$. For every $\alpha > 0$ and every subset B of Ω, the property $\mathcal{Q}^{m,\alpha}_\psi(B)$ is said to hold if

$$|B|^{\alpha+\psi(|B|)} \leq m\big(B\big) \leq |B|^{\alpha-\psi(|B|)}. \tag{8}$$

Heuristically, $\mathcal{Q}_\psi^{m,\alpha}(B)$ holds when $m(B) \sim |B|^\alpha$. For $\gamma > 0$, $\alpha > 0$, $\xi \geq 1$ and $0 < \rho < 1$, we define for every $n \in \mathbb{N}$ the balls

$$B_n^\gamma = B(x_n, \lambda_n^\gamma) \tag{9}$$

$$\text{and} \quad B_{n,\xi}(m,\rho,\alpha,\psi) = \begin{cases} B_n^\xi & \text{if } \mathcal{Q}_\psi^{m,\alpha}(B_n^\rho) \text{ holds} \\ \emptyset & \text{otherwise.} \end{cases} \tag{10}$$

Essentially $B_{n,\xi}(m,\rho,\alpha,\psi)$ is non-empty and equal to a contracted ball B_n^ξ as soon as a condition (8) on the larger ball B_n^ρ holds.

For every $x \in \Omega$ and $\mathbf{k} \in \{-1,0,1\}^d$ the lower and upper Hölder exponents of a positive Borel measure m at neighborhood $\mathbf{k}$ of x are respectively defined by

$$\underline{\alpha}_m^{\mathbf{k}}(x) = \liminf_{j\to+\infty} \frac{\log_c m(I_{j,\mathbf{k}_{j,x}^c+\mathbf{k}}(x))}{j} \quad \text{and} \quad \overline{\alpha}_m^{\mathbf{k}}(x) = \limsup_{j\to+\infty} \frac{\log_c m(I_{j,\mathbf{k}_{j,x}^c+\mathbf{k}}(x))}{j}.$$

When $\underline{\alpha}_m^{\mathbf{k}}(x) = \overline{\alpha}_m^{\mathbf{k}}(x)$, their common value is denoted $\alpha_m^{\mathbf{k}}(x)$ and called the Hölder exponent at neighborhood $\mathbf{k}$ of x. One defines

$$\widetilde{E}_\alpha^m = \Big\{ x \in \Omega : \forall\, \mathbf{k} \in \{-1,0,1\}^d,\ \alpha_m^{\mathbf{k}}(x) = \alpha \Big\}. \tag{11}$$

For $x \in \Omega$, we also need the notion of upper Hölder exponent (the counterpart of the exponent h_m defined in (4))

$$\overline{h}_m(x) = \limsup_{r\to 0+} \frac{\log m(B(x,r))}{\log |B(x,r)|} = \limsup_{j\to+\infty} \frac{\log m(B(x,c^{-j}))}{\log |B(x,c^{-j})|}.$$

The reader can easily check that one has $h_m(x) = \min_{\mathbf{k}\in\{-1,0,1\}^d} \underline{\alpha}_m^{\mathbf{k}}(x)$, and $\overline{h}_m(x) \leq \min_{\mathbf{k}\in\{-1,0,1\}^d} \overline{\alpha}_m^{\mathbf{k}}(x)$.

The scaling function, or L^q-spectrum, associated with a measure m is needed to invoke the multifractal formalism developed in [13]. For every integer $c \geq 2$, this function is defined by

$$\tau_{m,c} :\ q \mapsto \liminf_{j\to\infty} -\frac{1}{j} \log_c \Big(\sum_{\mathbf{k}\in\{0,\ldots,c^j-1\}^d} m(I_{j,\mathbf{k}}^c)^q \Big). \tag{12}$$

Since $\operatorname{supp}(m) = \Omega = [0,1]^d$, $\tau_{m,c}$ actually does not depend on the integer $c \geq 2$, and is consequently denoted by τ_m. It is a concave and non-decreasing mapping. The Legendre transform of τ_m at $\alpha \in \mathbb{R}_+$, denoted by τ_m^*, is defined by

$$\tau_m^* :\ \alpha \mapsto \inf_{q\in\mathbb{R}} \big(\alpha q - \tau_m(q)\big) \in \mathbb{R} \cup \{-\infty\}. \tag{13}$$

2.2. Irreducible elements of $\{x_n : n \geq 0\}$.

Let us denote $S = \{x_n : n \geq 0\}$.

Definition 2.1. *Let $y \in S$ and let $n_y = \min\{n : y = x_n\}$. The point x_{n_y} is called the irreducible form of y. If $p \geq 0$ is such that $n_{x_p} = p$, x_p is said to be irreducible.*

Notice that since $(\lambda_n)_{n\geq 1}$ is non-increasing, one has $\lambda_{n_y} \geq \lambda_n$ for all n such that $x_n = y$. This notion of irreducibility coincides with the usual notion of irreducibility when, in dimension 1, $\{(x_n,\lambda_n)\}_n$ is a sequence taking values in the set of rational pairs $\{(p/q, 1/q^2) : q \geq 1,\ 0 \leq p \leq q\}$.

From now on, for $x \in \Omega$, we simply write ξ_x instead of $\deg(x)$. The proof of the following proposition is immediate and left to the reader.

Proposition 2.2. *For every $x \in \Omega$ which verifies $\xi_x < \infty$, one has (remember that we have set $\xi_x = \deg(x)$)*

$$\xi_x = \lim_{n\to\infty} \sup_{y\in S,\ n_y \geq n} \frac{\log \|x-y\|_\infty}{\log \lambda_{n_y}}.$$

Notice that if the x_n's are pairwise distinct, all of them are irreducible.

2.3. Conditions on μ.

Definition 2.3. Property P1(μ): *There exists a constant M such that*

$$\forall x \in \Omega,\ \forall r > 0,\ \mu\big(B(x,r)\big) \geq r^M.$$

The role of property **P2**$(\mu, \rho, \{(x_n, \lambda_n)\}, h)$ is to enable us to get a lower bound for the Hausdorff dimension of sets of the form $\limsup_{n\to\infty} B_{n,\xi_n}(\mu, \rho, h, \psi)$, where $h > 0$ is a positive exponent, $(\xi_n)_{n\geq 0} \in [1,\infty)^{\mathbb{N}}$, and where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-decreasing continuous function such that $\psi(0) = 0$.

Definition 2.4. Property P2$(\mu, \rho, \{(x_n, \lambda_n)\}, h)$ and heterogeneous ubiquity: *There exists a non-decreasing continuous function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\psi(0) = 0$ and for every $\xi \geq 1$, one can find a non-decreasing sequence $(\xi_n)_{n\geq 0}$ converging to ξ and a positive Borel measure $m_{\rho,\xi}$ with the following properties (the balls $B_{n,\xi_n}(\mu, \rho, h, \psi)$ are defined in (10)): Let*

(14)

$$T = \limsup_{n\to\infty} B_{n,\xi_n}(\mu, \rho, h, \psi) \text{ and } d(\tau_\mu^*(h), \rho, \xi) = \min\left(\frac{d(1-\rho)+\rho\tau_\mu^*(h)}{\xi}, \tau_\mu^*(h)\right)$$

One has $m_{\rho,\xi}(T) > 0$, and for every Borel set E satisfying $\dim E < d(\tau_\mu^*(h), \rho, \xi)$, one has $m_{\rho,\xi}(E) = 0$.

In particular, $\dim(T) \geq d(\tau_\mu^*(h), \rho, \xi)$.

Heuristically, T is a subset of Ω which contains points which are approximated at rate $\xi \geq 1$ by some points x_n (relatively to λ_n), those points x_n being selected according to the value of the μ-measure of the ball $B_n^\rho = B(x_n, \lambda_n^\rho)$.

Property **P2**$(\mu, \rho, \{(x_n, \lambda_n)\}, h)$ is shown to be satisfied for large classes of systems and statistically self-similar measures in [11] (see also Section 5). It is also shown in [11] that if $\{(x_n, \lambda_n)\}_n$ is weakly redundant, then $\dim(T) \leq d(\tau_\mu^*(h), \rho, \xi)$.

Remark that, under the weak redundancy assumption, a saturation phenomenon occurs when $\rho < 1$, in the following sense. As long as $\xi \leq \xi_c = \frac{d(1-\rho)+\rho\tau_\mu^*(h)}{\tau_\mu^*(h)}$, one has $\dim T = \tau_\mu^*(h)$, while when $\xi \geq \xi_c$, the dimension of the T starts to decrease and $\dim T = \frac{d(1-\rho)+\rho\tau_\mu^*(h)}{\xi}$. This fact plays a fundamental role in the computation of the multifractal spectrum of ν_ρ when $\rho < 1$ in Theorem 2.9.

Definition 2.5. Property P3(μ, h): *There exists a positive Borel measure μ_h with $supp(\mu) = \Omega$, $\mu_h(\widetilde{E}_h^\mu) > 0$, and $\mu_h(E) = 0$ for every Borel set E such that $\dim E < \tau_\mu^*(h)$ (the level set $\widetilde{E}_h^\mu$ is defined by (11)).*

Property **P3**(μ, h) implies the validity of the multifractal formalism for μ at h in the sense that in this case $\dim E_\mu^h = \tau_\mu^*(h)$ (one always has $\dim E_\mu^h \leq \tau_\mu^*(h)$, see Proposition 4.4).

2.4. **Statement of the main result.** Let us consider a measure μ on $\mathbb{R}^d$ such that $\text{supp}(\mu) = \Omega$ and a weakly redundant system $\{(x_n, \lambda_n)\}_n$. Let $\rho \in (0, 1]$ be the dilation parameter introduced in the beginning of Section 1 in order to defined the measure ν_ρ in (6).
Two last properties are needed.

Definition 2.6. Property P4: *The system $\{(x_n, \lambda_n)\}_n$ satisfies*

$$\Omega \setminus \{x_n : n \geq 0\} \subset \limsup_{n\to\infty} B(x_n, \lambda_n/2). \tag{15}$$

As in (1), (15) ensures that $\deg(x) = \xi_x \geq 1$ for all $x \in \Omega \setminus \{x_n : \ n \geq 0\}$.

Remark 2.7. *It is important to notice that by construction (15) also holds if the family $\{(x_n, \lambda_n)\}_n$ is restricted to the pairs such that x_n is irreducible.*

Definition 2.8. Property P5(μ): *Let $q_c(\mu)$ be the (critical) real number defined by*

$$q_c(\mu) = \inf\{q : \ \tau_\mu(q) = 0\}.$$

Due to the definition of τ_μ (12), one always has $q_c(\mu) \in (0, 1]$. We assume that

$$q_c(\mu) = 1.$$

P5(μ) is satisfied for example as soon as μ is an atomless measure such that τ_μ^* is negative in a neighborhood of 0^+ (this implies that the lower Hölder exponents of μ range in an interval isolated from 0). This situation occurs for many classes of measures obtained by using a multiplicative scheme (see again Section 5 for examples). We then define

$$h_c(\mu) = \tau'_\mu(q_c(\mu)^-) = \tau'_\mu(1^-).$$

Now, let us emphasize that in the next result, **P1(μ)** allows to obtain an upper bound for the multifractal spectrum d_{ν_ρ} of ν_ρ, and that **P2($\mu, \rho, \{(x_n, \lambda_n)\}, h$)** and **P3($\mu, h$)** are necessary to be able to get a lower bound for d_{ν_ρ}.

Theorem 2.9. *Let μ be a positive Borel measure on $\Omega = [0, 1]^d$, and let $\{(x_n, \lambda_n)\}_n$ be a weakly redundant system in Ω. Let $\rho \in (0, 1]$, and ν_ρ be the measure obtained in (6). Assume that* **P1(μ)**, **P4** *and* **P5(μ)** *together hold.*
Case $\rho = 1$.

(1) *If $h_c(\mu) > 0$, then $d_{\nu_1}(h) \leq h$ for every $h \in [0, h_c(\mu)]$.*
Moreover, if **P2($\mu, 1, \{(x_n, \lambda_n)\}, h_c(\mu)$)** *holds, then $d_{\nu_1}(h) = h$ for every $h \in [0, h_c(\mu)]$.*
(2) *If $h \geq h_c(\mu)$, then $d_{\nu_1}(h) \leq \tau_\mu^*(h)$ if $\tau_\mu^*(h) \geq 0$, and $E_h^{\nu_1} = \emptyset$ if $\tau_\mu^*(h) < 0$.*
Moreover, if **P3(μ, h)** *holds then $d_{\nu_1}(h) = \tau_\mu^*(h)$.*

Case $\rho < 1$.

(1) *If $h_c(\mu) > 0$, then $d_{\nu_\rho}(h) \leq h$ for every $h \in [0, h_c(\mu)]$.*
Moreover, if **P2($\mu, \rho, \{(x_n, \lambda_n)\}, h_c(\mu)$)** *holds, then $d_{\nu_\rho}(h) = h$ for every $h \in [0, h_c(\mu)]$.*
(2) *Consider the exponent*

$$h_\rho(\mu) = d(1 - \rho) + \rho\tau'_\mu(0^+). \tag{16}$$

If $h \in (h_c(\mu), h_\rho(\mu))$ (this interval is non-empty if and only if $\tau'_\mu(0^+) > d$, or equivalently $h_c(\mu) < d$), there exists a unique $\alpha = \alpha(h) \in (h_c(\mu), \tau'_\mu(0^+))$

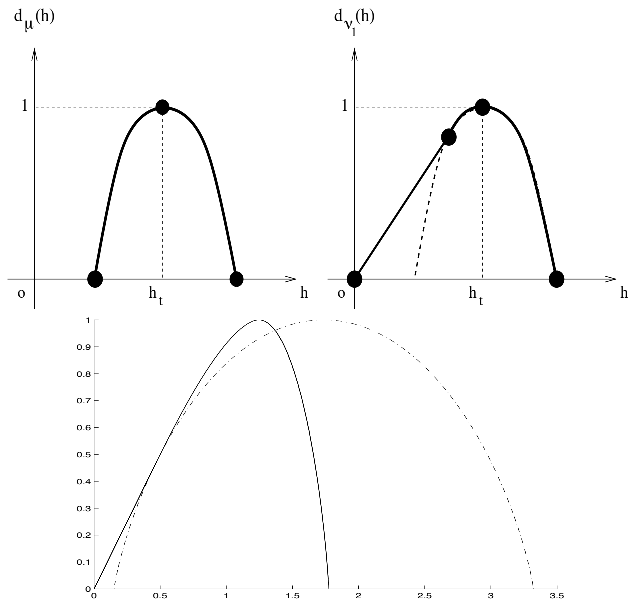

FIGURE 1. **Top left:** Typical Hausdorff spectrum d_μ of a random self-similar measure μ. **Top right:** Hausdorff spectrum d_{ν_1} of the corresponding sum of Dirac masses ν_1 with $\rho = 1$. **Bottom:** Hausdorff spectrum d_μ of the binomial measure μ with $p_0 = 1/10$ and $p_1 = 9/10$ (dashed graph), and Hausdorff spectrum $d_{\nu_{1/3}}$ of its associated sum of Dirac masses with $\rho = 1/3$ (plain graph). The multifractal spectrum of $\nu_{1/3}$ is highly asymmetric.

such that $\tau_\mu^*(\alpha) = h\frac{d(1-\rho)+\rho\tau_\mu^*(\alpha)}{d(1-\rho)+\rho\alpha}$. *One has* $\alpha(h) > h$ *and* $d_{\nu_\rho}(h) \leq \tau_\mu^*\big(\alpha(h)\big)$.

Moreover, if $\mathbf{P2(}\mu, \rho, \{(x_n, \lambda_n)\}, \alpha(h)\mathbf{)}$ *holds then* $d_{\nu_\rho}(h) = \tau_\mu^*\big(\alpha(h)\big)$.

(3) *If* $h \geq h_\rho(\mu)$, *then* $d_{\nu_\rho}(h) \leq \tau_\mu^*(\beta(h))$, *where*

$$\beta(h) = \frac{h - d(1-\rho)}{\rho}. \tag{17}$$

Moreover, $E_h^{\nu_\rho} = \emptyset$ *if* $\tau_\mu^*(\beta(h)) < 0$.

Finally, if $\mathbf{P3(}\mu, \beta(h)\mathbf{)}$ *holds, then* $d_{\nu_\rho}(h) = \tau_\mu^*(\beta(h))$.

Consequently, as claimed in the simpler context of Theorem 1.3, under some assumptions on μ and $\{(x_n, \lambda_n)\}$, when $h_c(\mu) > 0$, the multifractal spectrum of ν_ρ is composed of two parts: a linear part (starting at $(0, 0)$) when h is smaller than the critical value $h_c(\mu)$, and then a concave part when $h \geq h_c(\mu)$. This is a mixture between the linear shape obtained in the homogeneous case (2) in Theorem 1.1 and the classical strictly concave spectrum of statistically self-similar measures obeying some multifractal formalism [13, 30].

The heterogeneous ubiquity is responsible for the linear part of the spectrum, and when $\rho < 1$, it is also responsible for the value of the spectrum on the interval $(h_c(\mu), h_\rho(\mu))$. We are able to prove that $h \mapsto \tau_\mu^*(\alpha(h))$ is concave on the interval $(h_c(\mu), h_\rho(\mu))$ only when $d = 1$.

Section 5 provides examples illustrating Theorem 2.9, including the case $h_c(\mu) = 0$. The proof of Theorem 2.9 begins in Section 3, where the sets $E_h^{\nu_\rho}$ are characterized in terms of the measure μ in Theorem 3.2. The proof ends in Section 4, where Theorem 3.2 is used to find an upper bound and a lower bound for $\dim E_h^{\nu_\rho}$.

It is natural to ask whether some multifractal formalism is satisfied by ν_ρ or not. This question is discussed in Section 6.

3. Local regularity of ν_ρ and level sets of h_{ν_ρ}

Let μ be a positive Borel measure on $\Omega = [0,1]^d$ with $\mathrm{supp}(\mu) = \Omega$ and c an integer ≥ 2. A weakly redundant system $\{(x_n, \lambda_n)\}_n$ is also fixed. Let us remark that using the weak redundancy assumption on $\{(x_n, \lambda_n)\}_n$, if $y \in S$, then (remember Definition 2.1 for the value of n_y)

$$\frac{\lambda_{n_y}^{d(1-\rho)}\mu(B(y,\lambda_{n_y}^\rho))}{c_{n_y}|\log \lambda_{n_y}|^2} \leq \nu_\rho(\{y\}) \leq 4\,\frac{\lambda_{n_y}^{d(1-\rho)}\mu(B(y,\lambda_{n_y}^\rho))}{|\log \lambda_{n_y}|}. \tag{18}$$

Definition 3.1. *Let $\alpha \geq 0$, $\xi \geq 1$ be two real numbers. Let $\varepsilon > 0$. For every point $x \in \Omega$, the property $\mathcal{P}(\rho,\alpha,\xi,\varepsilon)$ is said to hold at x if there exist $\eta < \varepsilon$ and an infinite number of irreducible points $y \in S$ such that*

$$\lambda_{n_y}^{\rho(\alpha+\eta)} \leq \mu\big(B(y,\lambda_{n_y}^\rho)\big) \leq \lambda_{n_y}^{\rho(\alpha-\eta)} \text{ and } \|x-y\|_\infty \leq \lambda_{n_y}^{\xi-\eta}. \tag{19}$$

For $h \geq 0$, let

$$F_{h,\rho} = \left\{ x \in \Omega : \left\{ \begin{array}{c} \forall \varepsilon > 0,\ \exists\, \alpha \geq 0,\ \xi \geq 1 \text{ such that} \\ \frac{d(1-\rho)+\rho\alpha}{\xi} \leq h+\varepsilon \text{ and } \mathcal{P}(\rho,\alpha,\xi,\varepsilon) \text{ holds at } x \end{array} \right. \right\} \tag{20}$$

$$G_{h,\rho} = F_{h,\rho} \bigcup \{x \in \Omega : h_\mu(x) \leq \max(\beta(h), h)\}. \tag{21}$$

It is immediate that for any $0 \leq h \leq h'$, $F_{h,\rho} \subset F_{h',\rho}$ and $G_{h,\rho} \subset G_{h',\rho}$

The following result exhibits, for every $h > 0$, two sets A_h and B_h such that $A_h \subset E_h^{\nu_\rho} \subset B_h$. The sets A_h and B_h are used to find respectively a lower bound and an upper bound for $\dim E_h^{\nu_\rho}$.

Theorem 3.2. *Assume that* **P1(μ)** *holds, and let $h > 0$.*
One has $A_h \subset E_h^{\nu_\rho} \subset B_h$, where

(1) *If $h < h_\rho(\mu)$, then*

$$\begin{cases} A_h = \left(\widetilde{E}^\mu_{\max(h,\beta(h))} \cap \{x \in \Omega : \xi_x = 1\}\right) \bigcup \left(F_{h,\rho} \setminus \bigcup_{h'<h} G_{h',\rho}\right), \\ B_h = G_{h,\rho} \setminus \bigcup_{h'<h} F_{h',\rho}. \end{cases}$$

(2) *If $h \geq h_\rho(\mu)$, then*

$$\begin{cases} A_h = \widetilde{E}^\mu_{\beta(h)} \cap \{x \in \Omega : \xi_x = 1\}, \\ B_h = \bigcup_{h' \geq \beta(h)} \overline{E}^\mu_{h'}. \end{cases}$$

Theorem 3.2 is a consequence of next Propositions 3.3, 3.5 and 3.6.

Proposition 3.3. *Let $x \in \Omega \setminus S = \Omega \setminus \{x_n : n \geq 0\}$. If $\xi_x < +\infty$, then*

$$\min\left(\frac{d(1-\rho)+\rho h_\mu(x)}{\xi_x}, h_\mu(x)\right) \leq h_{\nu_\rho}(x) \leq \frac{d(1-\rho)+\rho \overline{h}_\mu(x)}{\xi_x}. \tag{22}$$

Moreover, if **P1(μ)** *holds and $\xi_x = +\infty$, then $h_{\nu_\rho}(x) = 0$.*

Proof. We denote $\alpha = \underline{h}_\mu(x)$, $\beta = \overline{h}_\mu(x)$. Let $\varepsilon > 0$, and assume that $\xi_x < +\infty$.

• Let us first find an upper bound for $h_{\nu_\rho}(x)$.
Let us begin with the case $\xi_x > 1$. By definition of ξ_x and by **P4**, there exists an infinite number of irreducible points $x_p \in S$ such that $\|x - x_p\|_\infty \le \lambda_{n_{x_p}}^{\xi_x - \varepsilon}/2$. Let x_p be such a point. One has by (18)

$$\nu_\rho\Big(B\big(x, \lambda_{n_{x_p}}^{\xi_x-\varepsilon}/2\big)\Big) \ge \nu_\rho\left(\{x_p\}\right) \ge |\log \lambda_{n_{x_p}}|^{-2} c_{n_{x_p}}^{-1} \lambda_{n_{x_p}}^{d(1-\rho)} \mu\Big(B\big(x_p, \lambda_{n_{x_p}}^{\rho}\big)\Big).$$

By definition of $\beta = \overline{h}_\mu(x)$, there exists a scale J_1 such that

$$0 < r < 2^{-J_1} \text{ implies } \mu\left(B\left(x, r\right)\right) \ge r^{\beta+\varepsilon}.$$

By construction one has $B(x, \lambda_{n_{x_p}}^{\rho}/2) \subset B(x_p, \lambda_{n_{x_p}}^{\rho})$. Consequently, $\mu\big(B(x_p, \lambda_{n_{x_p}}^{\rho})\big) \ge \mu\big(B(x, \lambda_{n_{x_p}}^{\rho}/2)\big) \ge (\lambda_{n_{x_p}}^{\rho}/2)^{\beta+\varepsilon}$ when n_{x_p} is large enough. Hence

$$\nu_\rho\Big(B\big(x, \lambda_{n_{x_p}}^{\xi_x-\varepsilon}/2\big)\Big) \ge 2^{-(\beta+\varepsilon)} c_{n_{x_p}}^{-1} |\log \lambda_{n_{x_p}}|^{-2} \lambda_{n_{x_p}}^{d(1-\rho)+\rho(\beta+\varepsilon))}. \tag{23}$$

Because of the weak redundancy of $\{(x_n, \lambda_n)\}_n$, one has $\log c_{n_{x_p}} = o(|\log \lambda_{n_{x_p}}|)$. Thus one obtains from (23) that $h_{\nu_\rho}(x) \le \frac{d(1-\rho)+\rho(\beta+\varepsilon)}{\xi_x - \varepsilon}$. This remains true for every $\varepsilon > 0$. Hence $h_{\nu_\rho}(x) \le \frac{d(1-\rho)+\rho\overline{h}_\mu(x)}{\xi_x}$.

If $\xi_x = 1$, $\|x - x_p\|_\infty \le \lambda_{n_{x_p}}^{\xi_x-\varepsilon}/2$ above is replaced by $\|x - x_p\|_\infty \le \lambda_{n_{x_p}}/2$, and the previous lines show that $h_{\nu_\rho}(x) \le d(1-\rho) + \rho\overline{h}_\mu(x)$. These lines also imply that if $\mathbf{P1}(\mu)$ holds and $\xi_x = +\infty$, then $h_{\nu_\rho}(x) = 0$.

• Let us now find a lower bound for $h_{\nu_\rho}(x)$. This part is more delicate to obtain. By definition of $\alpha = \underline{h}_\mu(x)$, there exists J_2 such that

$$0 < r < 2^{-J_2} \text{ implies } \mu\left(B\left(x, r\right)\right) \le r^{\alpha-\varepsilon}. \tag{24}$$

By Proposition 2.2, there exists a scale J_3 such that for every $y \in S$ such that $\lambda_{n_y} \le 2^{-J_3}$, $\|x - y\|_\infty \ge \lambda_{n_y}^{\xi_x+\varepsilon}$. One can also take $J_4 \ge J_3$ such that if $y \in S$ and $\lambda_{n_y} \le 2^{-J_4}$ then $\forall\, y' \in S$ such that $\lambda_{n_{y'}} \ge 2^{-J_3}$, $y' \notin B\left(x, \lambda_{n_y}\right)$.

In the rest of the proof, we often use the weak redundancy property (Definition 1.2) and the decomposition of the set T_j into pairwise disjoint subsets $T_j = \bigcup_{i=1}^{N_j} T_{j,i}$, for some sequence $(N_j)_{j\ge 0}$ such that $\log N_j = o(j)$.

Let $j_0 \ge 3\xi_x \frac{\max(J_2, J_4)}{\rho}$. Let us find a upper bound $\nu_\rho(B(x, 2^{-j_0}))$. One decomposes $\nu_\rho\left(B\left(x, 2^{-j_0}\right)\right)$ into c, where

$$u_j(x) = N_j^{-1} \sum_{n\in T_j} \mathbb{1}_{B(x,2^{-j_0})}\left(x_n\right) \frac{\mu(B(x_n, \lambda_n^{\rho}))}{\lambda_n^{-d(1-\rho)} |\log \lambda_n|^2}.$$

We split the sum $\sum_{j\ge 0} u_j(x)$ into three terms which are studied independently below.

• **Estimate of** $\sum_{0\le j<j_0} u_j(x)$**:** Suppose that this sum is not equal to 0, and let $n_0 = \min\Big\{n : x_n \in \bigcup_{j<j_0} \{x_p : p \in T_j \text{ and } x_p \in B(x, 2^{-j_0})\}\Big\}$. One obviously has $n_0 = n_{x_{n_0}}$, i.e. x_{n_0} is its own irreducible form. This implies that $\lambda_{n_0}^{\xi_x+\varepsilon} \le$

$\|x-x_{n_0}\|_\infty \leq 2^{-j_0} < \lambda_{n_0}$. More generally, using that $j_0 \geq 3\xi_x \frac{\max(J_2,J_4)}{\rho}$, for every $y \in \bigcup_{j_0/(\xi_x+\varepsilon)\leq j<j_0} \{x_n \in B(x,2^{-j_0}) : n \in T_j\}$,

$$\lambda_{n_y}^{\xi_x+\varepsilon} \leq \|x-y\|_\infty \leq 2^{-j_0} < \lambda_{n_y}. \tag{25}$$

The previous remark yields

$$\sum_{0\leq j<j_0} u_j(x) = \sum_{j_0/(\xi_x+\varepsilon)\leq j<j_0} u_j(x). \tag{26}$$

Let $j_0/(\xi_x+\varepsilon) \leq j < j_0$ and $1 \leq i \leq N_j$. Using the weak redundancy property $T_j = \bigcup_{i=1}^{N_j} T_{j,i}$, there is at most one integer element n of $T_{j,i}$ such that $x_n \in B(x_0, 2^{-j})$. Hence there are at most N_j non-zero terms in $u_j(x)$.

For every $j_0/(\xi_x+\varepsilon) \leq j < j_0$ and $n \in T_j$ such that $x_n \in B(x, 2^{-j})$, one has $B(x_n, \lambda_n^\rho) \subset B(x, 2\cdot 2^{-j_0\rho/(\xi_x+\varepsilon)})$. Thus each non-zero term in $u_j(x)$ is bounded by $N_j^{-1} 2^{-d(1-\rho)j} \mu\big(B(x, 2\cdot 2^{-j_0\rho/(\xi_x+\varepsilon)})\big)$. Consequently, using (24),

$$\begin{aligned}\sum_{j_0/(\xi_x+\varepsilon)\leq j<j_0} u_j(x) &\leq \mu\big(B(x,2\cdot 2^{-j_0\rho/(\xi_x+\varepsilon)}\big) \sum_{j_0/(\xi_x+\varepsilon)\leq j<j_0} 2^{-d(1-\rho)j} \\ &\leq \mu\big(B(x,2\cdot 2^{-j_0\rho/(\xi_x+\varepsilon)}\big) j_0 2^{-d(1-\rho)j_0/(\xi_x+\varepsilon)} \\ &\leq 2^{\alpha-\varepsilon} j_0 2^{-j_0\frac{d(1-\rho)+\rho(\alpha-\varepsilon))}{\xi_x+\varepsilon}}.\end{aligned}$$

• **Estimate of $\sum_{j_0\leq j<j_0/\rho} u_j(x)$:** Let j be an integer belonging to $[j_0, j_0/\rho]$. Fix $1 \leq i \leq N_j$. Applying again the weak redundancy property and the fact that the (disjoint) balls $B(x_n,\lambda_n)$, $n \in T_{j,i}$, have a radius ranging between $2^{-(j+1)}$ and 2^{-j}, there exists a universal constant C such that the number of elements n of $T_{j,i}$ such that $x_n \in B(x, 2^{-j_0})$ is bounded by $P2^{d(j-j_0)}$. Moreover, for each such n, one has $\mu(B(x_n,\lambda_n^\rho)) \leq \mu(B(x, 2\cdot 2^{-j\rho}))$. Using again (24), this yields

$$\begin{aligned}\sum_{j_0\leq j<j_0/\rho} u_j(x) &\leq \sum_{j_0\leq j<j_0/\rho} C\, 2^{d(j-j_0)} 2^{-d(1-\rho)j} 2^{\alpha-\varepsilon} 2^{-j\rho(\alpha-\varepsilon)} \\ &\leq C' 2^{-dj_0} \sum_{j_0\leq j<j_0/\rho} 2^{j\rho(d-(\alpha-\varepsilon))}.\end{aligned}$$

If $\alpha > d$ and ε is small enough, the last sum is bounded by $C2^{-dj_0} 2^{j_0\rho(d-(\alpha-\varepsilon))} \leq C2^{-j_0(d(1-\rho)+\rho(\alpha-\varepsilon))}$.

If $\alpha \leq d$, the sum is bounded by $C2^{-j_0(\alpha-\varepsilon)}$.

• **Estimate of $\sum_{j>j_0/\rho} u_j(x)$:** The following result is needed.

Lemma 3.4. *There exists a universal constant C having the following property.*

For every $y \in [0,1]^d$, for every $j_0 \geq 0$ and for every $j > j_0/\rho$, let $\{B(y_p, r_p)\}_p$ be any family of pairwise disjoint balls of radii r_p ranging from $2^{-(j+1)}$ to 2^{-j} such that $y_p \in B(y, 2^{-j_0}) \cap [0,1]^d$.

There exists an integer $N \leq C2^{d(1-\rho)j}$ such that the family $\{B(y_p, r_p^\rho)\}_p$ can be partitioned into N subfamilies of pairwise disjoint balls.

Proof. We use the dyadic cubes $I^2_{j,\mathbf{k}}$ of Section 2.1. Let $j \geq j_0/\rho$ and $j_\rho = [j\rho]$.

If $\mathbf{k} \in \{0,1\}^d$, let $A_{\mathbf{k}} = \Big\{p : \exists\ \mathbf{k}' \in \{0, \ldots 2^{j_\rho-1}-1\}^d,\ y_p \in I^2_{j_\rho, 2\mathbf{k}'+\mathbf{k}}\Big\}$. Then let $\mathcal{B}_{\mathbf{k}} = \big\{B(y_p, r_p^\rho) :\ p \in A_{\mathbf{k}}\big\}$. By construction, the 2^d sets $\mathcal{B}_{\mathbf{k}}$ are pairwise disjoint and their union is equal to $\{B(y_p, r_p^\rho)\}_p$. Now, it remains to show that for each

$\mathbf{k} \in \{0,1\}^d$, if $\mathcal{B}_{\mathbf{k}} \neq \emptyset$, this set can be partitioned into at most $C2^{d(1-\rho)j}$ subfamilies of pairwise disjoint balls, for some universal constant C.

On the one hand, in every cube of the form $I^2_{j_\rho, 2\mathbf{k}'+\mathbf{k}}$, $\mathbf{k}' \in \{0, \dots 2^{j_\rho - 1} - 1\}^d$, there are at most $C2^{d(1-\rho)j}$ balls $B(y_p, r_p)$, where C is a constant depending only on d. Indeed, the volume of any of the disjoint balls $B(y_p, r_p)$, $p \in \mathcal{A}_{\mathbf{k}}$, is greater than $2^{-d(j+1)}$, and the volume of the cube $I^2_{j_\rho, 2\mathbf{k}'+\mathbf{k}}$ is 2^{-dj_ρ}.

On the other hand, if two points $y_{p_1} \in I^2_{j_\rho, 2\mathbf{k}'_1+\mathbf{k}}$ and $y_{p_2} \in I^2_{j_\rho, 2\mathbf{k}'_2+\mathbf{k}}$ with $\mathbf{k}'_1, \mathbf{k}'_2 \in \{0, \dots, 2^{j_\rho-1} - 1\}^d$ and $\mathbf{k}'_1 \neq \mathbf{k}'_2$, then $B(y_{p_1}, r^\rho_{p_1}) \cap B(y_{p_2}, r^\rho_{p_2}) = \emptyset$. This makes it possible to split $\mathcal{B}_{\mathbf{k}}$ suitably. □

Let $j > j_0/\rho$. This time, if $1 \leq i \leq N_j$, for every $n \in T_{j,i}$, one sees that the ball $B(x_n, \lambda_n^\rho)$ has a diameter smaller than $2 \cdot 2^{-j_0}$. Thus if the corresponding point x_n belongs to $B(x, 2^{-j_0})$, then $B(x_n, \lambda_n^\rho)$ is contained in $\widetilde{B} = B(x, 3 \cdot 2^{-j_0})$.

Lemma 3.4 can be applied with $y = x$ and to $\{B(y_p, r_p)\} = \{B(x_n, \lambda_n^\rho) : n \in T_{j,i} \text{ and } x_n \in B(x, 2^{-j_0})\}$. This family can be split into $N_{j,i,\rho} \leq P2^{jd(1-\rho)}$ families of disjoint balls $\mathcal{B}_{j,i,k}$, and $\bigcup_{B \in \mathcal{B}_{j,i,k}} B \subset \widetilde{B}$. Hence

$$\begin{aligned} u_j(x) &= N_j^{-1} \sum_{n \in T_j} \mathbb{1}_{B(x, 2^{-j_0})}(x_n) \frac{\mu(B(x_n, \lambda_n^\rho))}{\lambda_n^{-d(1-\rho)} |\log \lambda_n|^2} \\ &\leq \frac{2^{-jd(1-\rho)}}{j^2 \log^2(2)} N_j^{-1} \sum_{i=1}^{N_j} \sum_{k=1}^{N_{j,i,\rho}} \sum_{B \in \mathcal{B}_{i,j,k}} \mu(B) \leq Cj^{-2} \mu(\widetilde{B}), \end{aligned}$$

for another constant C. Consequently

$$\sum_{j > j_0/\rho} u_j(x) \leq Cj_0^{-1} \mu(\widetilde{B}) \leq P2^{-j_0(\alpha - \varepsilon)}.$$

As a consequence, if ε is small enough, for j_0 large enough:

- If $\alpha \leq d$, $\nu_\rho(B(x, 2^{-j_0})) \leq C\big(2^{-j_0 \frac{d(1-\rho)+\rho(\alpha-\varepsilon)}{\xi_x + \varepsilon}} + 2^{-j_0(\alpha-\varepsilon)} + 2^{-j_0(\alpha-\varepsilon)}\big)$.
- If $\alpha > d$, $\nu_\rho(B(x, 2^{-j_0})) \leq C\big(2^{-j_0 \frac{d(1-\rho)+\rho(\alpha-\varepsilon)}{\xi_x + \varepsilon}} + 2^{-j_0(d(1-\rho)+\rho(\alpha-\varepsilon))} + 2^{-j_0(\alpha-\varepsilon)}\big)$.

In all these cases (and remarking that $d(1-\rho) + \rho\alpha$ is always greater than $\frac{d(1-\rho)+\rho\alpha}{\xi_x}$), letting ε go to zero gives Proposition 3.3. □

Proposition 3.5. *If $x \in F_{h,\rho}$ for some $h \geq 0$, then one has $h_{\nu_\rho}(x) \leq h$.*

Proof. Let $\varepsilon > 0$, $\alpha \geq 0$ and $\xi \geq 1$ such that $\frac{d(1-\rho)+\rho\alpha}{\xi} \leq h + \varepsilon$ and $\mathcal{P}(\rho, \alpha, \xi, \varepsilon)$ holds at x. Let $y \in S$ be such that (19) holds. One has $\nu_\rho\big(B\big(x, 2\lambda_{n_y}^{\xi-\eta}\big)\big) \geq \nu_\rho(\{y\})$. The proof ends as the one of the upper bound for $h_{\nu_\rho}(x)$ in Proposition 3.3. □

The next proposition entails the important role of the sets $G_{h,\rho}$, which contain the determinant information to study the Hausdorff dimensions of the level sets $E_h^{\nu_\rho}$.

Proposition 3.6. *Assume that $\mathbf{P1}(\mu)$ holds. Then for every $0 < h \leq h_\rho(\mu)$, one has $E_h^{\nu_\rho} \subset G_{h,\rho}$.*

Proof. Let $x \in E_h^{\nu_\rho}$. Let us denote $\alpha_x = h_\mu(x)$, $h_\rho = \max\big(\beta(h), h\big)$, and let $\varepsilon > 0$. One assumes that $\alpha_x > h_\rho$, and one wants to show that $x \in F_{h,\rho}$.

By definition of $E_h^{\nu_\rho}$, there exist a sequence of integers $(j_p)_{p\geq 0}$ such that

$$\nu_\rho\left(B\left(x, 2^{-j_p}\right)\right) \geq 2^{-j_p(h+\varepsilon/2)}. \tag{27}$$

We fix the integers J_2 and J_4 as in the Proposition 3.3's proof. Using results obtained in items (i), (ii) and (iii) of Proposition3.3, if j_p is large enough, then

$$\nu_\rho\left(B\left(x, 2^{-j_p}\right)\right) \leq C(2^{-j_p(d(1-\rho)+\rho(\alpha_x-\varepsilon))} + 2^{-j_p(\alpha_x-\varepsilon)}) + \sum_{j_p/(\xi_x+\varepsilon)\leq j<j_p} u_j(x). \tag{28}$$

Since $\alpha_x > h_\rho$, ε can be chosen small enough so that $\min(d(1-\rho)+\rho(\alpha_x-\varepsilon), \alpha_x-\varepsilon) > h+2\varepsilon$. Using (27) and (28), one gets for j_p large enough

$$2^{-j_p(h+\varepsilon)} \leq \sum_{j_p/(\xi_x+\varepsilon)\leq j<j_p} N_j^{-1} \sum_{n\in T_j} \mathbb{1}_{B(x,2^{-j_p})}(x_n) \frac{\mu(B(x_n,\lambda_n^\rho))}{\lambda_n^{-d(1-\rho)}|\log\lambda_n|^2}.$$

Let n_p be minimal in the subset of $\bigcup_{j_p/(\xi_x+\varepsilon)\leq j<j_p} T_j$ made of the integers k such that

$$\frac{\mu(B(x_k,\lambda_k^\rho))}{\lambda_k^{-d(1-\rho)}|\log\lambda_k|^2} = \max_{n\in\bigcup_{j_p/(\xi_x+\varepsilon)\leq j<j_p} T_j} \mathbb{1}_{B(x,2^{-j_p})}(x_n) \frac{\mu(B(x_n,\lambda_n^\rho))}{\lambda_n^{-d(1-\rho)}|\log\lambda_n|^2},$$

and notice that if $x_n = x_{n'}$ with $n' > n$ one has $\frac{\mu(B(x_n,\lambda_n^\rho))}{\lambda_n^{-d(1-\rho)}|\log\lambda_n|^2} \geq \frac{\mu(B(x_{n'},\lambda_{n'}^\rho))}{\lambda_{n'}^{-d(1-\rho)}|\log\lambda_{n'}|^2}$. This fact and our choice of the integer J_4 together imply that the point x_{n_p} is irreducible. Combining the remarks of item **1.** of Proposition 3.3 and the definition of n_p, one gets

$$2^{-j_p(h+\varepsilon)} \leq \frac{\mu(B(x_{n_p},\lambda_{n_p}^\rho))}{\lambda_{n_p}^{-d(1-\rho)}|\log\lambda_{n_p}|^2}. \tag{29}$$

Let us write $\left|x-x_{n_p}\right| = \lambda_{n_p}^{\xi_p} \leq \lambda_{n_p}$ with $\xi_p \geq 1$, and $\mu\big(B(x_{n_p},\lambda_{n_p}^\rho)\big) = \lambda_{n_p}^{\rho\alpha_p}$ for some $\alpha_p \geq 0$. Using (29) and (25), this exponent α_p satisfies

$$2^{-j_p(h+\varepsilon)} \leq \lambda_{n_p}^{d(1-\rho)+\rho\alpha_p} \leq 2^{-j_p\frac{d(1-\rho)+\rho\alpha_p}{\xi_p}},$$

which implies $\frac{1-\rho+\rho\alpha_p}{\xi_p} \leq h+\varepsilon$.

Performing this analysis for every p large enough gives us an infinite sequence of irreducible points x_{n_p} that satisfy $\left|x-x_{n_p}\right| = \lambda_{n_p}^{\xi_p}$ and $\mu\big(B(x_{n_p},\lambda_{n_p}^\rho)\big) = \lambda_{n_p}^{\rho\alpha_p}$ with $(\xi_p,\alpha_p) \in [1,\xi_x+1]\times[0,M]$. Indeed, by construction $\xi_p \leq \xi_x+1$ for p large enough, and M is the upper bound implied by the verification of condition $\mathbf{P1}(\mu)$. Thus, up to a subsequence, (ξ_p,α_p) converges toward $(\xi,\alpha) \in [1,\xi_x+1]\times[0,M]$.

Since $\frac{1-\rho+\rho\alpha_p}{\xi_p} \leq h+\varepsilon$ for every p, the couple (ξ,α) also satisfies this inequality, and by construction x satisfies $\mathcal{P}(\rho,\alpha,\xi,\varepsilon)$. Finally $x \in F_{h,\rho}$. □

4. Computation of the spectrum of ν_ρ: Proof of Theorem 2.9

As above, the system $\{(x_n,\lambda_n)\}_n$ is always supposed to be weakly redundant.

4.1. **Preliminary results.**

4.1.1. **Some Large Deviation estimates**.

Definition 4.1. *The function $\tau_{\mu,\rho}$ is defined by*

$$\tau_{\mu,\rho}(q) = \liminf_{j\to+\infty} -\frac{1}{j}\log_2 \sum_{n\in T_j} \mu\big(B(x_n,\lambda_n^\rho)\big)^q. \tag{30}$$

It is established in [11] that if the system $\{(x_n,\lambda_n)\}_n$ is weakly redundant, then

$$\forall\, q\in\mathbb{R},\ \tau_{\mu,\rho}(q)\geq d(1-\rho)+\rho\tau_\mu(q). \tag{31}$$

The following result is a consequence of some Tchernoff inequalities and (31).

Lemma 4.2. *For every $0\leq\beta\leq\alpha$ and for every $\varepsilon>0$, there exists a scale J such that $j\geq J$ implies*

$$\frac{\log_2\#\left\{n\in T_j:\lambda_n^{-\rho\alpha}\leq\mu\big(B(x_n,\lambda_n^\rho)\big)\leq\lambda_n^{-\rho\beta}\right\}}{j}\leq d(1-\rho)+\rho\sup_{\beta\leq\alpha'\leq\alpha}\tau_\mu^*(\alpha')+\varepsilon.$$

Tchernoff inequalities also yield the following lemma, in which c is any integer ≥ 2.

Lemma 4.3. *Let $0\leq\beta\leq\alpha$, $M>1$ and $\varepsilon>0$. There exists a scale J such that*

$$j\geq J\Rightarrow\begin{cases} j^{-1}\log_c\left(\#\left\{\mathbf{k}:M^{-1}c^{-j\alpha}\leq\mu(I_{j,\mathbf{k}}^c)\right\}\right)\leq\tau_\mu^*(\alpha)+\varepsilon & \text{if } \alpha\leq\tau_\mu'(0^+),\\ j^{-1}\log_c\left(\#\left\{\mathbf{k}:\mu(I_{j,\mathbf{k}}^c)\leq Mc^{-j\beta}\right\}\right)\leq\tau_\mu^*(\beta)+\varepsilon & \text{if } \beta\geq\tau_\mu'(0^+).\end{cases}$$

4.1.2. **Upper bound for the Hausdorff dimension of union of level sets**. The following proposition is a consequence of [13], [30], [25] and [5]. The results concerning the set $\widetilde{E}_\alpha^\mu$ defined in Section 2.1 are valid in any basis $c\geq 2$.

Proposition 4.4. *Let $\alpha\geq 0$.*

1. *If $\tau_\mu^*(\alpha)<0$, then $\widetilde{E}_\alpha^\mu=E_\alpha^\mu=\emptyset$.*
2. *One has $\tilde{d}_\mu(\alpha)\leq\tau_\mu^*(\alpha)$.*
3. *If $\alpha\in[0,\tau_\mu'(0^+)]$, then $\dim\bigcup_{\alpha'\leq\alpha}E_{\alpha'}^\mu\leq\tau_\mu^*(\alpha)$.*
4. *If $\alpha\geq\tau_\mu'(0^+)$, then $\dim\bigcup_{\alpha'\geq\alpha}\overline{E}_{\alpha'}^\mu\leq\tau_\mu^*(\alpha)$.*

4.1.3. **Upper bound for the Hausdorff dimension of some limsup sets**. Let us introduce, for $0\leq\beta\leq\alpha$ and $\xi\geq 1$, the sets

$$S_\mu(\rho,\alpha,\beta,\xi)=\bigcap_{N\geq 0}\ \bigcup_{n\geq N:\lambda_n^{\rho\alpha}\leq\mu\left(B(x_n,\lambda_n^\rho)\right)\leq\lambda_n^{\rho\beta}} B(x_n,\lambda_n^\xi).$$

These sets are useful to find an upper bound for the spectrum of ν_ρ.

Lemma 4.5. *Let $\{(x_n,\lambda_n)\}_n$ be a weakly redundant system. For every $\rho\in(0,1]$, $0\leq\beta\leq\alpha$ and $\xi\geq 1$,*

$$\dim S_\mu(\rho,\alpha,\beta,\xi)\leq\min\Big(\sup_{\beta\leq\alpha'\leq\alpha}\tau_\mu^*(\alpha'),\frac{d(1-\rho)+\rho\sup_{\beta\leq\alpha'\leq\alpha}\tau_\mu^*(\alpha')}{\xi}\Big). \tag{32}$$

Proof. • First we show that $\dim S_{\rho,\alpha,\beta,\xi}\leq\sup_{\beta\leq\alpha'\leq\alpha}\tau_\mu^*(\alpha')$.

If $\beta\leq\tau_\mu'(0^+)\leq\alpha$, there is nothing to prove since $\sup_{\beta\leq\alpha'\leq\alpha}\tau_\mu^*(\alpha')=d$.

Let $j\geq 0$ and $n\in T_j$ such that $\mu\big(B(x_n,\lambda_n^\rho)\big)\in[\lambda_n^{\rho\alpha},\lambda_n^{\rho\beta}]$. Let $j_\rho=[(j+1)\rho]+1$. By construction, there exists a dyadic cube I_ρ of generation j_ρ such that $I_\rho\cap B(x_n,\lambda_n^\rho)\neq\emptyset$ and $\mu(I_\rho)\geq 2^{-j_\rho\alpha}/4^d$. The cube I_ρ can be written $I_{j_\rho,\mathbf{k}_\rho}^2$ (defined in Section 2.1) for some multi-integer $\mathbf{k}_\rho\in\{0,\dots,2^{j_\rho}-1\}^d$. Also, there exists $\mathbf{k}_\rho'\in\{-8,..,0,..,8\}^d$ such that $I_{j_\rho,\mathbf{k}_\rho+\mathbf{k}_\rho'}^2\subset B(x_n,\lambda_n^\rho)$, and thus $\mu\big(I_{j_\rho,\mathbf{k}_\rho+\mathbf{k}_\rho'}^2\big)\leq$

$2^{-j_\rho\beta}$. Moreover, one has $B(x_n,\lambda_n^\rho)\subset\bigcup_{\mathbf{k}\in\{-8,\dots,0,\dots,8\}^d} I^2_{j_\rho,\mathbf{k}_\rho+\mathbf{k}}$. It follows that

$$(33)\quad S_\mu(\rho,\alpha,\beta,\xi)\ \subset\ \limsup_{j\to\infty}\bigcup_{\mathbf{k}\in\{0,\dots,2^j-1\}^d:\ \frac{2^{-j\alpha}}{4}\le\mu(I^2_{j,\mathbf{k}})}\ \bigcup_{\mathbf{k'}\in\{-8,\dots,0,\dots,8\}^d} I^2_{j,\mathbf{k}+\mathbf{k'}}$$

$$(34)\quad S_\mu(\rho,\alpha,\beta,\xi)\ \subset\ \limsup_{j\to\infty}\bigcup_{\mathbf{k}\in\{0,\dots,2^j-1\}^d:\ \mu(I^2_{j,\mathbf{k}})\le 2^{-j\beta}}\ \bigcup_{\mathbf{k'}\in\{-16,\dots,0,\dots,16\}^d} I^2_{j,\mathbf{k}+\mathbf{k'}}.$$

Suppose that $\alpha\le\tau_\mu'(0^+)$. Then $\sup_{\beta\le\alpha'\le\alpha}\tau^*_\mu(\alpha')=\sup_{\alpha'\le\alpha}\tau^*_\mu(\alpha')=\tau^*_\mu(\alpha)$. Let $\varepsilon>0$. By Lemma 4.3, for j large enough,

$$\frac{\log\#\{\mathbf{k}:\frac{2^{-j\alpha}}{4}\le\mu(I^2_{j,\mathbf{k}})\}}{\log 2^j}\le d_\varepsilon,$$

where $d_\varepsilon=\tau^*_\mu(\alpha)+\varepsilon$. Let $\gamma>d_\varepsilon$. One then remarks that $\sum_{j\ge1}s_j<\infty$, where

$$s_j:=\sum_{k\in\{0,\dots,2^j-1\}^d:\ \frac{2^{-j\alpha}}{4}\le\mu(I^2_{j,\mathbf{k}}))\le 42^{-j\beta}}\ \sum_{\mathbf{k'}\in\{-8,\dots,0,\dots,8\}^d}|I^2_{j,\mathbf{k}+\mathbf{k'}}|^\gamma\le 17^d2^{jd_\varepsilon}2^{-j\gamma}.$$

Combining this with (33) yields the vanishing of the γ-dimensional Hausdorff measure of $S_\mu(\rho,\alpha,\beta,\xi)$. Hence for every $\varepsilon>0$, $\dim S_\mu(\rho,\alpha,\beta,\xi)\le d_\varepsilon$. Letting ε go to zero yields $\dim\, S_\mu(\rho,\alpha,\beta,\xi)\le\tau^*_\mu(\alpha)$.

The case $\tau_\mu'(0^+)<\beta\le\alpha$ is treated similarly by using (34).

• The upper bound $\dim\, S_\mu(\rho,\alpha,\beta,\xi)\le\big(d(1-\rho)+\rho\sup_{\beta\le\alpha'\le\alpha}\tau^*_\mu(\alpha')\big)/\xi$ is a simple consequence of the definition of $S_\mu(\rho,\alpha,\beta,\xi)$ and Lemma 4.2. □

4.2. **Upper bound for the multifractal spectrum of ν_ρ.** We first take care of the decreasing part of d_{ν_ρ}.

Proposition 4.6. *Assume that* $\mathbf{P1}(\mu)$ *holds. If* $h\ge h_\rho(\mu)$*, then* $\dim E_h^{\nu_\rho}\le\tau^*_\mu(\beta(h))$.

Proof. It is a consequence of Theorem 3.2(ii) and Proposition 4.4(4). □

The increasing part is more delicate to study.

Proposition 4.7. *Assume that* $\mathbf{P1}(\mu)$ *holds. Let* $0<h<h_\rho(\mu)$*. If* $h<d$*, then*

$$\dim E_h^{\nu_\rho}\le\max\Big(\tau^*_\mu(h),\sup_{\alpha\ge h}\min\Big(\tau^*_\mu(\alpha),h\frac{d(1-\rho)+\rho\tau^*_\mu(\alpha)}{d(1-\rho)+\rho\alpha}\Big)\Big),$$

and if $h\ge d$*, then*

$$\dim E_h^{\nu_\rho}\le\max\Big(\tau^*_\mu(\beta(h)),\sup_{\alpha\ge\beta(h)}\min\Big(\tau^*_\mu(\alpha),h\frac{d(1-\rho)+\rho\tau^*_\mu(\alpha)}{d(1-\rho)+\rho\alpha}\Big)\Big).$$

Proof. By Theorem 3.2(i), $E_h^{\nu_\rho}\subset F_{h,\rho}\bigcup\{x\in\Omega: h_\mu(x)\le\max(\beta(h),h)\}$.

• **Case** $h\in[d,h_\rho(\mu))$**:** Here $\max\big(h,\beta(h)\big)=\beta(h)$.

By Proposition 4.4, $\dim\,\{x\in\Omega:h_\mu(x)\le\beta(h)\}\le\tau^*_\mu(\beta(h))$. It remains us to find an upper bound for the Hausdorff dimension of $F_{h,\rho}$.

Let us fix $0<\varepsilon\le1$. For every $i\in\mathbb{N}$, let $\xi_i=1+i\varepsilon$ and let $\alpha_i=\xi_i\frac{(h+\varepsilon)-d(1-\rho)}{\rho}$.

Let $x\in F_{h,\rho}$. There exists a couple (α,ξ) and $\eta<\varepsilon$ such that (19) holds. Let i_x be the unique integer so that $\xi\in[\xi_{i_x},\xi_{i_x+1})$. By construction,

$$\alpha\le\frac{\xi(h+\varepsilon)-d(1-\rho)}{\rho}\le\alpha_{i_x}+\frac{\varepsilon(h+\varepsilon)}{\rho}.$$

So $\alpha+\varepsilon \leq \alpha_{i_x}+M_h\varepsilon$, where $M_h = 1+(h+1)/\rho$. Since (19) holds, this implies that $x \in S_\mu(\rho, \alpha_{i_x}+M_h\varepsilon, 0, \xi_i)$.

As a consequence, $F_{h,\rho} \subset \bigcup_{i\in\mathbb{N}} S_\mu(\rho, \widetilde{\alpha}_i, 0, \xi_i)$, where $\widetilde{\alpha}_i = \alpha_i + M_h\varepsilon$. This implies by Lemma 4.5 that

$$\begin{aligned}\dim F_{h,\rho} &\leq \sup_{i\in\mathbb{N}} \dim S_\mu(\rho,\widetilde{\alpha}_i,0,\xi_i)\\ &\leq \sup_{i\in\mathbb{N}} \min\Big(\sup_{\alpha'\leq\widetilde{\alpha}_i}\tau_\mu^*(\alpha'), \frac{d(1-\rho)+\rho\sup_{\alpha'\leq\widetilde{\alpha}_i}\tau_\mu^*(\alpha')}{\xi_i}\Big).\end{aligned}$$

One always has $(h+\varepsilon)\xi_i = d(1-\rho)+\rho\alpha_i$, hence

$$\dim F_{h,\rho} \leq \sup_{i\in\mathbb{N}} \min\Big(\sup_{\alpha'\leq\widetilde{\alpha}_i}\tau_\mu^*(\alpha'), (h+\varepsilon)\,\frac{d(1-\rho)+\rho\sup_{\alpha'\leq\widetilde{\alpha}_i}\tau_\mu^*(\alpha')}{d(1-\rho)+\rho\alpha_i}\Big).$$

This upper bound remains true when $\varepsilon\to 0$. Then, using the continuity of the Legendre transform τ_μ^* on the interval where it is finite and remembering that $\alpha_i \geq \beta(h)$, we get

$$\dim F_{h,\rho} \leq \sup_{\alpha\geq\frac{h-d(1-\rho)}{\rho}} \min\left(\sup_{\alpha'\leq\alpha}\tau_\mu^*(\alpha'), h\frac{d(1-\rho)+\rho\sup_{\alpha'\leq\alpha}\tau_\mu^*(\alpha')}{d(1-\rho)+\rho\alpha}\right).$$

The result follows from the fact that τ_μ^* is non-decreasing on $[0,\tau_\mu'(0^+)]$ and non-increasing on $[\tau_\mu'(0^+),\infty)$.

• **Case** $h<1$**:** The same lines of computations apply here, except that α_i has to be taken larger than or equal to h (instead of $\frac{h-(1-\rho)}{\rho}$). □

4.3. **Simplification of Proposition 4.7.** We denote $\alpha_{\min} = \inf\{\alpha : \tau_\mu^*(\alpha)\geq 0\}$ and $h_c = h_c(\mu)$. Remember that by $\mathbf{P5(\mu)}$ one has $q_c(\mu)=1$ and $\tau_\mu^*(h_c(\mu)) = h_c$. We also point out the fact that $h_c(\mu) = \tau_\mu'(0^+)$ if and only if $\tau_\mu'(0^+) = d$ (i.e. τ_μ is affine between 0 and 1), and the interval $(h_c(\mu), h_\rho(\mu))$ is then empty.

Proposition 4.8. (1) *For every $h\in[0,h_c(\mu)]$, the upper bound given by Proposition 4.7 equals h.*

(2) *Suppose that $\tau_\mu'(0^+) > d$. For every $h\in(h_c(\mu),h_\rho(\mu))$, there exists a unique $\alpha=\alpha(h)\in(h_c(\mu),h_\rho(\mu)]$ such that $h\frac{d(1-\rho)+\rho\tau_\mu^*(\alpha)}{d(1-\rho)+\rho\alpha} = \tau_\mu^*(\alpha)$, and the upper bound of Proposition 4.7 equals $\tau_\mu^*(\alpha(h))$.*

(3) *Suppose that $\tau_\mu'(0^+) > d$. Let*

$$\theta : h\mapsto \begin{cases} h & \text{if } h\in[0,h_c(\mu)]\\ \tau_\mu^*\big(\alpha(h)\big) & \text{if } h\in(h_c(\mu),h_\rho(\mu)]\end{cases} \quad \textit{and} \quad \overline{\theta} : h\mapsto \begin{cases}\tau_\mu^*(h) & \text{if } h\leq d\\ \tau_\mu^*(\beta(h)) & \text{if } h\in(d,h_\rho(\mu)].\end{cases}$$

If $\rho=1$ then $\theta=\overline{\theta}$ on $(h_c(\mu),h_\rho(\mu))$.
If $\rho<1$ then $\theta>\overline{\theta}$ on $(h_c(\mu),h_\rho(\mu))$, and θ is concave increasing if $d=1$.

Proof. Since $\tau_\mu^*(\alpha)\leq\alpha$ for every $\alpha\geq 0$, one has $\frac{d(1-\rho)+\rho\tau_\mu^*(\alpha)}{d(1-\rho)+\rho\alpha} \leq \frac{d(1-\rho)+\rho\tau_\mu^*(h_c)}{d(1-\rho)+\rho h_c} = 1$.

(i) Let $h\in[0,h_c]$. The mapping $\alpha\mapsto\tau_\mu^*(\alpha)$ is increasing on $[\alpha_{\min},h_c]$, and $\tau_\mu^*(h_c) = h_c \geq h = h\frac{1-\rho+\rho\tau_\mu^*(h_c)}{1-\rho+\rho h_c}$. Hence the upper bound of Proposition 4.7 is h.

(ii) Let $h\in(h_c,h_\rho(\mu))$. On one side, the function $f:\ \alpha\mapsto h\frac{d(1-\rho)+\rho\tau_\mu^*(\alpha)}{d(1-\rho)+\rho\alpha}$ is non-decreasing on $[0,h_c]$ and decreasing on $(h_c,\tau_\mu'(0^+)]$, and $f(h_c) = h > h_c$. On the other side, the mapping $\alpha\mapsto\tau_\mu^*(\alpha)$ is increasing on $[0,\tau_\mu'(0^+)]$ and $\tau_\mu^*(h_c) = h_c$.

Since the functions f and τ_μ^* are continuous, there exists a unique exponent $\alpha(h) \in (h_c, \tau_\mu'(0^+))$ such that $h\frac{1-\rho+\rho\tau_\mu^*(\alpha(h))}{1-\rho+\rho\alpha(h)} = \tau_\mu^*(\alpha(h))$.

If $\rho = 1$, then $\forall h \in (h_c, h_\rho(\mu))$, $\alpha(h) = h$ (and $\theta(h) = \overline{\theta}(h)$).

Let $\rho \in (0,1)$. To see that $\overline{\theta}(h) < \theta(h)$ on $(h_c, d]$, we show that $h < \alpha(h)$ on this interval. This is equivalent to $\frac{d(1-\rho)+\rho\tau_\mu^*(h)}{d(1-\rho)+\rho h} > \frac{\tau_\mu^*(h)}{h}$. This holds because when $h \geq h_c$ the mapping $h \mapsto \tau_\mu^*(h)/h$ is decreasing and the function $h \mapsto \frac{d(1-\rho)+\rho\tau_\mu^*(h)}{d(1-\rho)+\rho h}$ is increasing, and both functions coincide at h_c.

To see that $\overline{\theta}(h) < \theta(h)$ on $(d, h_\rho(\mu))$, we prove that $\beta(h) < \alpha(h)$ on this interval. This is equivalent to $d(1-\rho) + \rho\tau_\mu^*(\beta(h)) > \tau_\mu^*(\beta(h))$. This last inequality is true, since one has $\beta(h) < \tau_\mu'(0^+)$, which implies $\tau_\mu^*(\beta(h)) < d$.

Consequently, by definition of $\alpha(h)$, the upper bound of Proposition 4.7 equals $\theta(h) = \tau_\mu^*\big(\alpha(h)\big)$ for all $h \in (h_c, h_\rho(\mu))$.

(iii) To end the proof of item **3.**, it remains to show that the function θ is concave if $d = 1$. In fact it is easier to show that its inverse function θ^{-1} is convex.

Let us first assume that τ_μ^* is twice differentiable on $(h_c, h_\rho(\mu))$. The mapping $(\tau_\mu^*)^{-1}$ is well defined since τ_μ^* is continuous and strictly increasing when $h \leq \tau_\mu'(0^+)$. By definition, if $u = \theta(h)$ for some $h \in [h_c, h_\rho(\mu)]$, one has

$$\theta^{-1}(u) = u\frac{d(1-\rho) + \rho(\tau_\mu^*)^{-1}(u)}{d(1-\rho) + \rho u}.$$

A computation shows that on $[h_c, h_\rho(\mu))$, the second derivative of θ^{-1} is larger than a positive multiple of

$$d(1-\rho)\big[\big((\tau_\mu^*)^{-1}\big)'(u) - (\tau_\mu^*)^{-1}(u)\big] + \rho\big[u\big((\tau_\mu^*)^{-1}\big)'(u) - (\tau_\mu^*)^{-1}(u)\big].$$

This term is non-negative if $d = 1$. Indeed, on the interval $\theta\big([h_c, h_\rho(\mu))\big)$, the function $\big((\tau_\mu^*)^{-1}\big)'(u)$ is non-decreasing, $\big((\tau_\mu^*)^{-1}\big)'(u) \geq 1$, and $(\tau_\mu^*)^{-1}(h_c) = h_c$. So θ^{-1} is convex. The fact that θ remains concave near the intermediate point h_c is simply due to the fact that by construction $d_{\nu_\rho}(h) \leq h$.

The case where τ_μ^* is non-differentiable is obtained by approaching uniformly by above τ_μ^* on the interval $[h_c, h_\rho(\mu)]$ by increasing concave functions that are twice differentiable with derivatives smaller than or equal to 1. □

Remark: The upper bound given by Proposition 4.7 also applies when $h \geq h_\rho(\mu)$, since in this case

$$\sup_{\alpha \geq \beta(h)} \min\left(\tau_\mu^*(\alpha), h\frac{d(1-\rho) + \rho\tau_\mu^*(\alpha)}{d(1-\rho) + \rho\alpha}\right) = \tau_\mu^*(\beta(h)).$$

4.4. **Lower bound for the spectrum of ν_ρ.** Here again h_c denotes respectively $h_c(\mu)$. Let us take care of the decreasing part.

Proposition 4.9. *Let $h \geq h_\rho(\mu)$. If $\tau_\mu^*(\beta(h)) > 0$ and if* $\mathbf{P3}\big(\mu, \beta(h)\big)$ *is satisfied, then $d_{\nu_\rho}(h) \geq \tau_\mu^*(\beta(h))$.*

Proof. Let $m_{\beta(h)}$ be the measure given by $\mathbf{P3}(\mu, \beta(h))$. Using Theorem 3.2(ii), it is enough to show that $\dim \widetilde{E}^\mu_{\beta(h)} \cap \{x \in \Omega : \xi_x = 1\} \geq \tau_\mu^*(\beta(h))$, or equivalently by $\mathbf{P3}(\mu, \beta(h))$ that $m_{\beta(h)}(\widetilde{E}^\mu_{\beta(h)} \cap \{x \in \Omega : \xi_x > 1\}) = 0$.

Let $\varepsilon > 0$, $\eta > 0$ and $x \in \widetilde{E}^{\mu}_{\beta(h)} \setminus \{x_n : n \geq 1\}$. If $\xi_x > 1+\eta$, then there exist infinitely many $y \in \{x_n : n \geq 1\}$ such that $\|x-y\|_\infty \leq \lambda_{n_y}^{\xi_x - \eta/2} \leq \lambda_{n_y}$. By definition of $\widetilde{E}^{\mu}_{\beta(h)}$, this implies that for such a point y, if n_y is large enough, then $\lambda_{n_y}^{\beta(h)+\varepsilon} \leq \mu\big(B(y,\lambda_{n_y})\big) \leq \lambda_{n_y}^{\beta(h)-\varepsilon}$.

Thus, if $K_{\beta(h),1+\eta}$ is the set $\big(\widetilde{E}^{\mu}_{\beta(h)} \setminus \{x_n : n \geq 1\}\big) \cap \{x : \xi_x > 1+\eta\}$, then $K_{\beta(h),1+\eta} \subset S_\mu(1, \beta(h)+\varepsilon, \beta(h)-\varepsilon, 1+\eta/2)$. Lemma 4.5 yields that $\dim S_\mu(1,\beta(h)+\varepsilon, \beta(h)-\varepsilon, 1+\eta/2) \leq \sup_{\beta(h)-\varepsilon \leq \alpha' \leq \beta(h)+\varepsilon} \tau^*_\mu(\alpha')/(1+\eta/2)$. Since this is true for any $\varepsilon > 0$, the last inequality yields that $\dim K_{\beta(h),1+\eta} \leq \tau^*_\mu(\beta(h))/(1+\eta/2)$ and that $m_{\beta(h)}(K_{\beta(h),1+\eta}) = 0$ (by $\mathbf{P3}(\mu, \beta(h))$).

Since $\big(\widetilde{E}^{\mu}_{\beta(h)} \setminus \{x_n : n \geq 1\}\big) \cap \{x : \xi_x > 1\}$ is equal to the countable union $\bigcup_{i \geq 1} K_{\beta(h),1+1/i}$ and $\{x_n : n \geq 1\}$ is countable, $m_{\beta(h)}(\widetilde{E}^{\mu}_{\beta(h)} \cap \{x : \xi_x > 1\}) = 0$. □

The lower bound for the increasing part of the spectrum uses $\mathbf{P2}(\mu, \rho, \{(x_n,\lambda_n)\}, h)$.

Proposition 4.10. *Assume that $h_c(\mu) > 0$ and that $\mathbf{P2}(\mu, \rho, \{(x_n,\lambda_n)\}, h_c(\mu))$ holds. Then, for every $h \in [0, h_c(\mu)]$, one has $d_{\nu_\rho}(h) \geq h$.*

Proof. Let $h \in (0, h_c)$ and let $\xi_h = \frac{d(1-\rho)+\rho h_c}{h}$.

Condition $\mathbf{P2}(\mu, \rho, \{(x_n,\lambda_n)\}, h)$ yields that for some convenient choice of ψ and $(\xi_n)_{n\geq 0}$ (converging to ξ), $d(\tau^*_\mu(h_c), \rho, \xi)$ is a lower bound for the Hausdorff dimension of the set T defined in (14). In this case, $d(\tau^*_\mu(h_c), \rho, \xi) = h$. In addition, there exists a positive measure $m_{\rho,\xi}$ such that $m_{\rho,\xi}(T) > 0$ and $m_{\rho,\xi}(E) = 0$ for every set E such that $\dim E < h$. Moreover, by Theorem 3.2(i),

$$E^{\nu_\rho}_h \supset F_{h,\rho} \setminus \Big(\bigcup_{h'<h} G_{h',\rho}\Big) = F_{h,\rho} \setminus \bigcup_{i \geq [h^{-1}]+1} G_{h-1/i,\rho}.$$

One remarks that $T \subset F_{h,\rho}$. Indeed, every point of T satisfies the property $\mathcal{P}\big(\rho, h_c, \xi_h, \varepsilon\big)$ for all $\varepsilon > 0$ small enough.

The conclusion follows as in Proposition 4.9: Combining the estimates obtained in the proof of Proposition 4.7 one gets that $\dim F_{h-1/i,\rho} < h$ for every $i \geq [h^{-1}]+1$. Moreover, item (iii) of Proposition 4.4 yields $\dim \{x \in \Omega : h_\mu(x) \leq h - 1/i\} \leq \tau^*_\mu(h - \frac{1}{i}) < h$. Thus $m_{\rho,\xi}\left(\bigcup_{i \geq [h^{-1}]+1} G_{h-1/i,\rho}\right) = 0$, which yields $m_{\rho,\xi}(E^{\nu_\rho}_h) \geq m_{\rho,\xi}(T) > 0$. This finally implies $\dim E^{\nu_\rho}_h \geq h$.

Notice that the points x_n belong to $E^{\nu_\rho}_0$. □

Proposition 4.11. *Suppose that $\tau'_\mu(0^+) > d$. Let $h \in [h_c, h_\rho(\mu))$. Assume that either ($\mathbf{P2}(\mu, \rho, \{(x_n,\lambda_n)\}, \alpha(h))$ holds) or ($\rho = 1$ and $\mathbf{P3}(\mu, h)$ holds).*

Then $d_{\nu_\rho}(h) \geq \tau^*_\mu(\alpha(h))$.

Proof. Remember Theorem 3.2(i). In the case where $h = h_c$ or $\rho = 1$, when $\mathbf{P3}(\mu, h)$ holds, the proof is the same as the one of Proposition 4.9, since $\alpha(h) = h$ and $(\widetilde{E}^{\mu}_h \cap \{x : \xi_x = 1\}) \subset E_{h,\rho}$.

We suppose that $\rho < 1$ and that $\mathbf{P2}(\mu, \rho, \{(x_n,\lambda_n)\}, \alpha(h))$ holds. We proceed as in the proof of Proposition 4.10. Let $\xi_h = \frac{1-\rho+\rho\tau^*_\mu(\alpha(h))}{\tau^*_\mu(\alpha(h))}$. As above, $\mathbf{P2}(\mu, \rho, \{(x_n,\lambda_n)\}, h)$ implies that for some convenient choice of ψ and $(\xi_n)_{n\geq 0}$ converging to ξ, $d\big(\tau^*_\mu\big(\alpha(h)\big), \rho, \xi_h\big) = \tau^*_\mu(\alpha(h))$ is a lower bound for the Hausdorff dimension of the limsup-set T (14). Moreover, there is a positive measure m_{ρ,ξ_h}

such that $m_{\rho,\xi_h}(T) > 0$ and for every set E with $\dim E < \tau_\mu^*(\alpha(h))$, $m_{\rho,\xi_h}(E) = 0$. Finally, $T \subset F_{h,\rho}$, since every point of T satisfies $\mathcal{P}\big(\rho, \alpha(h), \xi_h, \varepsilon\big)$ for all $\varepsilon > 0$. Since $h \mapsto \tau_\mu^*\big(\alpha(h)\big)$ is increasing on $(h_c, h_\rho(\mu))$ and $\tau_\mu^*(h) < \tau_\mu^*(\alpha(h))$ (Proposition 4.8) the arguments used in the proof of Proposition 4.10 yield $\dim E_h^{\nu_\rho} \geq \tau_\mu^*(\alpha(h))$. □

5. Examples

In [11], several suitable systems and statistically self-similar measures illustrate the notion of heterogeneous ubiquity. Some of them are recalled here.

5.1. **Some suitable systems.**

• *Family of the b-adic numbers:* Fix b an integer ≥ 2. The family $\{(x_n, \lambda_n)\}_n = \{(\mathbf{k}b^{-j}, b^{-j})\}_{j\geq 0, k\in\{0,\ldots,b^j-1\}^d}$ obviously has all the good properties.

• *Family of the rational numbers:* The system $\big\{\big(\mathbf{p}/q, 2/q^{1+1/d}\big)\big\}$, where $q \in \mathbb{N}^*$, $\mathbf{p} = (p_1, p_2, \ldots, p_d) \in \{0, \ldots, q-1\}^d$ such that at least one fraction p_i/q is irreducible, is weakly redundant and satisfies **P4**.

• *Family of the $\{\{n\alpha\}, 1/n\}_{n\in\mathbb{N}}$:* Let α be an irrational number, such that its approximation degree by the family rational numbers above in the one dimensional case equals 1. The family $\{(\{n\alpha\}, 1/n)\}_{n\geq 1}$ is weakly redundant and satisfies **P4** ($\{x\}$ stands for the fractional part of x). The weak redundancy does not hold if the approximation degree of α is > 1.

• *Poisson point processes.* Let S be a Poisson point process with intensity $\ell \otimes \pi$ in the square $[0,1] \times (0,1]$, where ℓ denotes the Lebesgue measure on $[0,1]$ and π is a positive locally finite Borel measure on $(0,1]$ – see [24] for the construction of a Poisson point process –. Let us write this set S as $\{(y_n, r_n)\}_n$. Let $\beta = \inf\{\gamma : \int_{x\leq 1} x^\gamma\, \pi(dx) < \infty\}$. There exists a non-decreasing sequence β_n converging to β such that the system $\{(y_n, r_n^{\beta_n})\}_n$ is weakly redundant and satisfies **P4** – see [11] for more details –.

• *Random family based on uniformly distributed points:* Let $\{x_n\}_n$ be a sequence of points independently and uniformly distributed in $[0,1]^d$ and $\{\lambda_n\}_n$ a non-increasing sequence of positive numbers.

If $\limsup_{n\to+\infty}(\sum_{p=1}^n \lambda_p/2 - d\log n) = +\infty$ and $\limsup_{j\to\infty} j^{-1}\log_2 \# T_j = 1$ (T_j was defined in (5)), then the system $\{(x_n, \lambda_n)\}_n$ satisfies **P4**.

5.2. **Random self-similar measures satisfying conditions P1-3 for suitable systems.**

We mention two main classes of such measures. The first one is obtained in the thermodynamic formalism. The second one is made of limit of $[0,1]^d$-martingales, considered in the multiplicative chaos in the meaning of [21]. It is shown in [9, 10, 11] that these measures μ obey conditions **P1-3** for suitable systems $\{(x_n, \lambda_n)\}$ (including those of Section 5.1) when h ranges in the interval where $\tau_\mu^*(h) > 0$.

• *(Random) Gibbs measures.* These measures are obtained as fixed points of adjoints of Ruelle-Perron Fröbenius operators associated with an Hölder potential ϕ in the dynamical system $([0,1)^d, T)$, where $T(x) = cx \mod 1$ with $c \in \mathbb{N}\setminus\{0,1\}$ – see [33]–. Random counterparts of these measures are considered in [22, 17, 9].

• $[0,1]^d$*-martingales.* The first examples are the so-called *independent multiplicative cascades*, or *Mandelbrot martingales* introduced in [27] and then studied extensively in [27, 23, 18, 29, 28, 1, 15, 4, 5, 10]. They are a particular case of a wider class of $[0,1]^d$-martingales – see [7] – which satisfy condition $\mathbf{P2(}\mu,\rho,\{(x_n,\lambda_n)\},h\mathbf{)}$. This class also includes compound Poisson cascades introduced in [6], as well as their extension in [3], and other examples – see [7, 8, 10] for details –.

Notice that we are able to prove that $\mathbf{P2(}\mu,\rho,\{(x_n,\lambda_n)\},h\mathbf{)}$ holds true for these measures only in the case $\rho=1$ (see [10] for details).

• *The substitute to independent multiplicative cascade in the critical case of degeneracy.* These measures are constructed in [4] (using results of [26]) in order to study the end points of the multifractal spectrum of independent multiplicative cascades. A modification of the martingale used in the independent multiplicative cascades definition is involved in their construction. In our context, the interesting property of these measures is that they provide examples of measures μ such that $q_c(\mu)=1$ but $h_c(\mu)=0$ – see [10] – which satisfies condition $\mathbf{P2(}\mu,\rho,\{(x_n,\lambda_n)\},h\mathbf{)}$ when $\rho=1$.

6. Comments

When $\rho=1$ and conditions **P1-3** are satisfied for every $h\in\mathrm{supp}(d_\mu)$, the multifractal spectrum of ν_1 (associated with a suitable weakly redundant system) becomes the Legendre transform of the function $\widetilde{\tau_{\nu_1}}$ defined by

$$\widetilde{\tau_{\nu_1}}:q\mapsto\begin{cases}\tau_\mu(q) & \text{if } q\le q_c(\mu)\\ 0 & \text{if } q>q_c(\mu)\end{cases}.$$

The same lines of computation as in [8] apply here (up to a small correction due to the weak redundancy assumption). As a result, if ν_1 is the measure associated with this system and with a suitable measure μ, one has $\tau_{\nu_1}=\widetilde{\tau_{\nu_1}}$. This implies that the Hausdorff multifractal spectrum of ν_1 is the Legendre spectrum of its scaling function τ_{ν_1}. Consequently, ν_1 fulfills the (box [13] or centered [30]) multifractal formalisms for measures.

The context is quite different when $\rho<1$, and the computations are subtler. In a further work, we will see that if μ is non trivial, all the usual multifractal formalisms fail on the right of $h_c(\mu)$ in this case.

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