DETECTING AND CREATING OSCILLATIONS USING MULTIFRACTAL METHODS

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ABSTRACT. By comparing the Hausdorff multifractal spectrum with the large deviations spectrum of a given continuous function f, we find sufficient conditions ensuring that f possesses oscillating singularities.

Using a similar approach, we study the non-linear wavelet threshold operator which associates with any function $f = \sum_j \sum_k d_{j,k}\psi_{j,k} \in L^2(\mathbb{R})$ the function series f^t whose wavelet coefficients are $d^t_{j,k} = d_{j,k}\mathbf{1}_{|d_{j,k}|\geq 2^{-j\gamma}}$, for some fixed real number $\gamma > 0$. This operator creates a context propitious to have oscillating singularities. As a consequence, we prove that the series f^t may have a multifractal spectrum with a support larger than the one of f.

We exhibit an example of function $f \in L^2(\mathbb{R})$ such that the associated thresholded function series f^t effectively possesses oscillating singularities which were not present in the initial function f. This series f^t is a typical example of function with homogeneous non-concave multifractal spectrum and which does not satisfy the classical multifractal formalisms.

1. INTRODUCTION AND MOTIVATIONS

This work is devoted to the investigation of the relations, for a given function f, between the failure of the multifractal formalism for f and the presence of oscillating singularities of f. Detecting fast oscillations in functions, images, or processes, is an important issue in mathematics and physics. For instance, the presence of oscillating singularities often goes along with a failure of the multifractal formalism (for a generic example see [12]). Fast oscillations can also generate compression problems in signal and image processing.

The ideas developed all along this work are essentially inspired by classical multifractal theory. Multifractal analysis consists in studying the local regularity of measures, functions or distributions. Our work draws its interest from functions, and in this case, the local regularity of a function $f \in L^\infty_{loc}(\mathbb{R})$ at a given point x is often measured through the pointwise Hölder exponent $h_f(x)$, which yields a precise indicator of the wildness of the behavior of f around x.

Definition 1.1. Let $f \in L^\infty_{loc}(\mathbb{R})$ and $x_0 \in \mathbb{R}$. Let h be a positive real number with $h \notin \mathbb{N}$. The function f belongs to $C^h_{x_0}$ if and only if there exist a constant C and a polynomial P of degree smaller than $[h]$ such that

$$\forall\, x \in \mathbb{R},\ |f(x) - P(x - x_0)| \leq C|x - x_0|^h.$$

The *pointwise Hölder exponent* of f at x_0 is $h_f(x_0) = \sup\{h : f \in C^h_{x_0}\}$.

This exponent may vary rapidly from point to point, and $h_f(x_0)$ is thus very difficult to estimate for real data. This is why its level sets $E^f_h = \{x \in \text{Int}(I) : h_f(x) = h\}$ are rather

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considered. These global quantities E_h^f, for $h \geq 0$, contain precious information about the local behavior of the function f. Performing the multifractal analysis of f amounts to computing the Hausdorff multifractal spectrum of f, which is the mapping

$$h \geq 0 \to d_f(h) = \dim_H E_h^f, \tag{1.1}$$

where $\dim_H$ stands for the Hausdorff dimension. If $E_h^f = \emptyset$, then by convention one sets $d_f(h) = -\infty$. The knowledge of this multifractal spectrum and of the dimension of the sets E_h^f gives a geometric representation of the repartition of the singularities of the function f.

Wavelets play an important role in the following, and more generally in the field of multifractal analysis of functions. They are natural tools to study local regularity of functions. In particular, for a sufficiently regular function f (for instance if $f \in C^\varepsilon(\mathbb{R})$ for some $\varepsilon > 0$), wavelet coefficient's decay rates provide a characterization of the pointwise Hölder exponent of f at a given point x [11]. Wavelets also allow to study oscillating behaviors of functions around a point x, see [1, 10, 18, 15], Sections 2.1 and 2.3.

For sake of simplicity, we work in the one-dimensional context. There is no doubt that all the properties and results extend to the multi-dimensional case. We thus start from the decomposition of a continuous function $f \in L^2(\mathbb{R})$ on an orthogonal wavelet basis $\{\psi_{j,k}\}_{(j,k)\in\mathbb{Z}^2}$, constructed as in [14] for instance. The wavelet coefficients of f are defined as

$$d_{j,k} = 2^j \int_{\mathbb{R}} f(t)\psi_{j,k}(t)dt, \tag{1.2}$$

and one can write in $L^2(\mathbb{R})$ that $f = \sum_{(j,k)\in\mathbb{Z}^2} d_{j,k}\psi_{j,k}$.

The distribution (in the space-frequency domain) of the wavelet coefficients is fundamental in the global and local regularity analysis of a function. For instance, some *a priori* upper bounds for the Hausdorff multifractal spectrum of a function f can be found by estimating, when the scale $j \geq 1$ goes to infinity, the histograms of the wavelet coefficients of f of scale j (i.e. the set of coefficients $\{d_{j,k}\}_{k\in\mathbb{Z}}$ (see [2])).

In Section 3.2, we show that the comparison between the large deviations spectrum (based on wavelet coefficients and somehow related to the histograms of wavelet coefficients) and the multifractal spectrum sometimes allows to conclude to the presence of oscillating singularities for the function f. A consequence *a posteriori* is that, when the multifractal formalism is broken in the increasing part of the multifractal spectrum d_f of f, the function f possesses oscillating singularities. By combining this result with the works of Jaffard and his collaborators [12, 16, 8], one obtains a striking result: Almost all functions in suitable function spaces V possess oscillating singularities (see details in Section 3.2).

It is then natural to investigate the effect of a decorrelation between the large deviations and the multifractal spectrum. This is achieved by performing a threshold on wavelet coefficients as follows.

Let $f \in L^2(\mathbb{R})$ with its wavelet decomposition $f = \sum_{(j,k)\in\mathbb{Z}^2} d_{j,k}\psi_{j,k}$, and let $\gamma > 0$. The *threshold of order* γ of f is the wavelet series $f^t = \sum_{j,k} d^t_{j,k}\psi_{j,k} \in L^2(\mathbb{R})$, where

$$d^t_{j,k} = d_{j,k}\mathbf{1}_{|d_{j,k}|\geq 2^{-j\gamma}}.$$

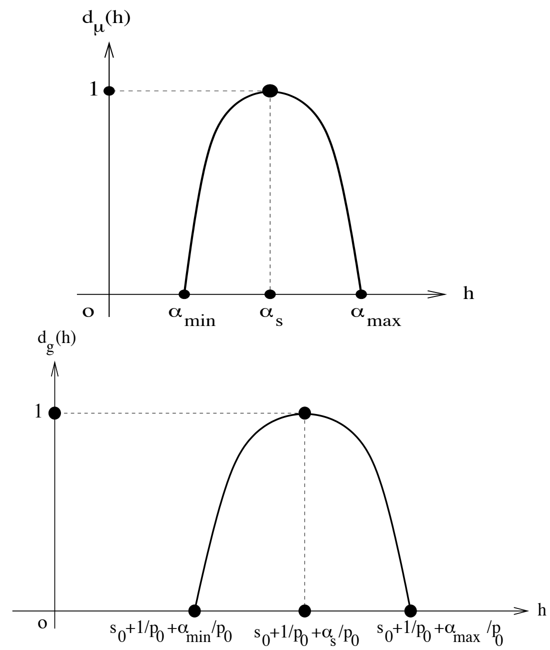

FIGURE 1. Typical multifractal spectrum of a binomial measure μ and of its associated wavelet series g.

We also introduce the *inverse threshold of order* γ which associates with the function $f \in L^2(\mathbb{R})$ the function series $f^{it} = \sum_{j,k} d^{it}_{j,k}\psi_{j,k}$, where

$$d^{it}_{j,k} = d_{j,k}\mathbf{1}_{|d_{j,k}| \leq 2^{-j\gamma}}.$$

A threshold usually keeps the largest wavelet coefficients, this is the reason why this transformation is called inverse threshold. The study of these operators appears to be fruitful from a multifractal point of view.

Threshold is a powerful method used in signal and image processing (see [6, 7] for the first introduction on wavelet thresholds). Indeed it is an easy way to extract and keep up the main information of a function (i.e. the largest wavelet coefficients) while giving up less important characteristics. This naturally leads to efficient compression methods. For discrete images, the classical *hard* threshold [6, 7] can certainly be compared to a threshold of order γ.

We show in Section 3.3 that the threshold of order γ introduces most of the time oscillating singularities for the “compressed” function f^t. We also prove that the support of multifractal spectrum of f^t may be different and larger than the one of f. Same properties hold true for the inverse threshold.

The rest of this work is devoted to the study of the action of both thresholds (regular and inverse) on the example inspired by [3] of a wavelet series g whose wavelet coefficients $\{d_{j,k}\}_{(j,k)\in\mathbb{Z}^2}$ are derived from a given positive measure μ in the following way. For sake of simplicity we focus on the binomial measure μ, but our results certainly apply to the class of random self-similar measures studied in [4] and to the wavelet series associated with these measures μ.

Let $0 < q_0 < 1/2$ and $q_1 = 1 - q_0$ be two real numbers, and let us consider the binomial measure μ with ratios (q_0, q_1) (its construction's scheme is recalled in Section 4). This measure μ satisfies the multifractal formalism for measures as defined in [5], and its multifractal spectrum is given by the following Legendre transform

$$d_\mu(\alpha) = \tau^*(\alpha) := \inf_{q\in\mathbb{R}} (q\alpha - \tau(q)), \tag{1.3}$$

where the function $\tau(q)$ is defined by

$$\tau(q) = \lim_{j\to+\infty} j^{-1} \log \sum_{k=0,\dots,2^j-1} \left[\mu([k2^{-j},(k+1)2^{-j}))\right]^q = -\log(q_0^q + q_1^q). \tag{1.4}$$

This spectrum is strictly concave, ranges in $[\alpha_{min} = -\log_2 q_1, \alpha_{max} = -\log_2 q_0]$. The exponent $\alpha_s = -\frac{1}{2}\log_2 q_0 q_1$ is the Lebesgue almost-sure exponent of μ (see Figure 1).

Starting from μ, using the same method as in [3], we construct a wavelet series g on $\mathbb{R}$ using the formula (where $s_0 > 0$, $p_0 > 0$, $s_0 - 1/p_0 \geq 0$)

$$g = \sum_{j\geq 1} \sum_{k=0}^{2^j-1} 2^{-j(s_0-1/p_0)} \mu\left([k2^{-j},(k+1)2^{-j})\right)^{1/p_0} \psi_{j,k}(x). \tag{1.5}$$

If the number of vanishing moments of the mother wavelet ψ is larger than $[\alpha_{\max}] + 1$, then (see [3]) the restriction of the function series g on $[0,1]$ has a multifractal spectrum (for functions) which is deduced from the one of μ through the formula

$$\dim_H(E_h^g \cap [0,1]) = d_g(h) = d_\mu\big(p_0(h - (s_0 - 1/p_0))\big).$$

Moreover, if restricted to $[0,1]$, the function series g satisfies the multifractal formalism for functions as defined in Section 2. This kind of result on wavelet cascades was actually expected since [1]. In [3], it is also shown that the wavelet series g does not contain any oscillating singularities (this holds true for any wavelet series f as soon as the wavelet coefficients of f are derived from a positive Borel measure as in (1.5)).

Applying the thresholds defined above to this function g yields two truncated (or "compressed") function series g^{it} and g^t. In Section 5, g^{it} and g^t are shown to have surprising multifractal spectra, see Theorems 5.1 and 5.2. These spectra are suprema of two mappings, and for well-chosen values of γ, they are not concave. In addition, oscillating singularities appear after applying a threshold on the wavelet coefficients, and we characterize their intensity and their repartition.

In both cases, the creation of oscillating singularities after applying a threshold on wavelet coefficients is somehow comparable to a Gibbs phenomenon in the sense that high frequencies appear around the singularities (isolated or not) in the compressed signals.

We end the paper by showing that, even though the multifractal formalism for functions is satisfied by the initial function series g, the truncated function series g^{it} and g^t do not obey the formalism any more. This illustrates the fragility of the multifractal formalism's validity.

2. LOCAL REGULARITY OF FUNCTIONS

2.1. **Wavelets and regularity exponents.** The functions we are interested in are supposed to possess some minimal global Hölder regularity. They can thus be decomposed on orthonormal wavelet bases. Let ψ be a wavelet in the Schwartz class, as constructed in [14] or [17]. The set of functions $\{\psi_{j,k} = \psi(2^j . - k)\}$, where $(j,k) \in \mathbb{Z}^2$, forms an orthogonal basis of $L^2(\mathbb{R})$. Any function $f \in L^2(\mathbb{R})$ can be written

$$f(x) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} d_{j,k}\psi_{j,k}(x), \tag{2.1}$$

where $d_{j,k}$ is the wavelet coefficient of f defined by (1.2). Note that we choose an L^∞ normalization for the wavelet coefficients.

Let $x_0 \in \mathbb{R}$. The pointwise Hölder exponent $h_f(x_0)$ of Definition 1.1 can be characterized by the decay rate of the wavelet coefficients around x_0. Indeed, if $f \in C^\varepsilon$ (uniformly) for some $\varepsilon > 0$ in a neighborhood of x_0, and if ψ has more than $[h_f(x_0)] + 1$ vanishing moments, then it is known (see [11]) that $h_f(x_0) = \liminf_{k2^{-j}\to x_0} \frac{\log |d_{j,k}|}{\log(2^{-j}+|x_0-k2^{-j}|)}$. **For the rest of the paper, ψ is fixed and is supposed to have enough vanishing moments so that the last equality holds at every** $x_0 \in (0,1)$. We do not go into details here, since we are going to use another characterization of the Hölder exponent via 2-microlocal analysis, see Section 2.3.

In the following, we focus on the local behavior of wavelet series $f \in L^\infty_{loc}(\mathbb{R})$ on an open bounded interval, which will typically be the interval $(0,1)$. Since multifractal analysis is concerned by what happens at infinitely small scales (i.e. when the integer j in the decomposition (2.1) is large), for any $x_0 \in (0,1)$, only the wavelet coefficients $d_{j,k}$ such that $j \geq 1$ and $k \in \{0,1,...,2^j-1\}$ may have an influence on the value of the pointwise Hölder exponents of f at x_0. This is the reason why, **without loss of generality for our results, we restrict our study to the functions** $f \in L^\infty_{loc}(\mathbb{R}) \cap L^2(\mathbb{R})$ **which can be written under the form**

$$f \in L^\infty_{loc}(\mathbb{R}) \cap L^2(\mathbb{R}) \text{ and } f(x) = \sum_{j\geq 1} \sum_{k=0}^{2^j-1} d_{j,k}\psi_{j,k}(x). \tag{2.2}$$

We also assume that **the mother wavelet ψ is** $C^\infty(\mathbb{R})$. Hence, if f has (2.2) for wavelet decomposition, for any $x_0 \in \mathbb{R}\backslash[0,1]$, f is C^∞ at x_0. Notice that if ψ is compactly supported, the local regularity at $x_0 \in \mathbb{R}\backslash[0,1]$ of a wavelet series f verifying (2.2) is ruled by the regularity of the wavelet ψ. In conclusion, due to our choices, **we obtain results for the multifractal spectrum of the restriction of the wavelet series f to the interval** $(0,1)$.

The exponent $h_f(x_0)$ is a measure of the behavior of a function f around a point x_0. But it does not distinguish, for example, at the point 0 the two functions $x \to |x|^\gamma$ and $x \to |x|^\gamma \sin(1/|x|^\beta)$. This second function is called a chirp at 0, and some infinitely fast oscillations occur around 0, in opposition to what happens at 0 for the first function, which is called a cusp.

Several methods ([1], [20]) have been introduced to detect and characterize oscillating singularities, i.e. infinitely fast oscillations around a point. For the rest of this work, we focus on the *oscillating exponent* introduced in [1], and denoted β_f. The advantage of this exponent is that it has a tractable wavelet and 2-microlocal characterization (see Section 2.3). This exponent β_f is obtained by considering what happens when an infinitesimal integration is performed:

Definition 2.1. Let $f \in L^\infty(\mathbb{R})$, and $x_0 \in \mathbb{R}$. We denote by $h_t(x_0)$ the pointwise Hölder exponent of a fractional primitive of order t of f at x_0. Then

$$\beta_f(x_0) = \Big(\frac{\partial}{\partial t} h_t(x_0)\Big)_{t=0^+} - 1.$$

One easily checks that $\beta_f(0)=0$ for the cusp, and $\beta_f(0)=\beta$ for the chirp. A function f is said to have an *oscillating singularity* at x_0 if $\beta_f(x_0)>0$.

2.2. **Various spectra of singularities for functions.** We consider three different spectra for a function f which can be decomposed into (2.2).

• The first one is the classical Hausdorff multifractal spectrum defined by

$$h \mapsto d_f(h) = \dim_H(E_h^f \cap (0,1)).$$

It measures the Hausdorff dimension of $E_h^f \cap (0,1)$, i.e. the size of the level sets of the pointwise Hölder exponent h_f of f restricted to $(0,1)$. Of course the Hausdorff multifractal spectrum does not depend on the choice of the wavelet basis $\{\psi_{j,k}\}_{j,k}$, provided that ψ is regular enough and has enough vanishing moments (which holds true in our case). Recall that, due to our choice for the wavelet series f, f is C^∞ at every $x\in\mathbb{R}\backslash[0,1]$.

• The second spectrum associated with a function $f\in$ satisfying (2.2) is called the Legendre spectrum. For such a function f, let us define the scaling function

$$\forall\, p\in\mathbb{R},\ \ \xi_f(p) = 1+\liminf_{j\to\infty} -j^{-1}\log_2 \sum_{k=0}^{2^j-1} |d_{j,k}|^p\ ,$$

where by convention $0^p=0\ \forall p$. This mapping ξ_f is concave and non-decreasing. The scaling function ξ_f actually does not depend on the choice of the wavelet ψ when $p>0$. Indeed, in this case an alternative definition of ξ_f is given in terms of Besov spaces: $\xi_f(p)=\sup\left\{u : f\in B^{u/p,\infty}_{p,loc}((0,1))\right\}$. Jaffard ([12] for instance) established the following general upper bound for $d_f(h)$ (when f has some minimal uniform regularity, which is our case in the sequel)

$$\forall\, h\geq 0,\ d_f(h)\leq \inf_{p\geq p_c}(ph-\xi_f(p)+1), \tag{2.3}$$

where p_c is the unique real number such that $\xi_f(p_c)=1$. Since [12], better upper-bounds have been found by replacing Besov spaces by the more general oscillation spaces, see [13], but this is not our purpose here.

The *Legendre spectrum* of a function f satisfying (2.2) is the Legendre transform $(\xi_f-1)^*$ of ξ_f-1:

$$(\xi_f-1)^* : h\geq 0 \mapsto \inf_{p\in\mathbb{R}}(ph-\xi_f(p)+1).$$

• Eventually, the *large deviations* spectrum $\tilde{d}_f$ of a function is related on the asymptotic histogram of wavelet coefficients, see [2] for a complete study of this spectrum. It is also relatively easy to estimate, in practical cases.

Definition 2.2. Let f satisfying (2.2), and let $\varepsilon>0$. For every $j\geq 1$ and $k\in\{0,\dots,,2^j-1\}$, let $h_{j,k}=-j^{-1}\log_2|d_{j,k}|$ (we set $h_{j,k}=+\infty$ if $d_{j,k}=0$). We set

$$N_j^\varepsilon(h) = \#\left\{k\in\{0,\dots,2^j\} : |h_{j,k}-h|\leq\varepsilon\right\}. \tag{2.4}$$

and $\tilde{d}^\varepsilon(h)=\limsup_{j\to+\infty} j^{-1}\log_2 N_j^\varepsilon(h)$.

The *large deviations spectrum* $\tilde{d}_f(h)$ is defined as the mapping $\tilde{d}_f(h)=\lim_{\varepsilon\to 0}\tilde{d}^\varepsilon(h)$.

The large deviations spectrum clearly depends on the choice of the wavelet ψ. While one always has $\tilde{d}_f(h)\leq(\xi_f-1)^*(h)$, there is no general relationship between $\tilde{d}_f$ and d_f. The examples we later consider illustrate this statement.

Definition 2.3. A function f satisfying (2.2) is said to obey the multifractal formalism at $h\geq 0$ if $d_f(h)=(\xi_f-1)^*(h)$.

2.3. **2-microlocal analysis of functions.** The notion of 2-microlocal spectrum developed in [15] is essential in the following study of local regularity. For every scale j and every exponent $\rho \in [0,1]$, $k_{j,\rho}$ denotes the integer $k_{j,\rho} = [2^{j(1-\rho)}]$.

Definition 2.4. Let f be a function satisfying (2.2). For any given x_0, define

- $\chi^0_{x_0} : (0,1) \to \mathbb{R}^+ \cup \{+\infty\}$,

$$\chi^0_{x_0}(\varepsilon) = \sup\{\gamma : \exists \delta,\ K,\ \forall \beta \leq \varepsilon,\ 2^{-j} \leq \delta \Rightarrow |d_{j,[2^j x_0 \pm k_{j,\beta}]}| \leq K 2^{-j\gamma}\}.$$

- For any given $\rho \in (0,1)$, $\chi^\rho_{x_0} : (0, \min(\rho, 1-\rho)) \to \mathbb{R}^+ \cup \{+\infty\}$,

$$\chi^\rho_{x_0}(\varepsilon) = \sup\{\gamma : \exists \delta,\ K,\ \forall \beta \in [\rho - \varepsilon, \rho + \varepsilon],\ 2^{-j} \leq \delta \Rightarrow |d_{j,[2^j x_0 \pm k_{j,\beta}]}| \leq K 2^{-j\gamma}\}.$$

- $\chi^1_{x_0} : (0,1) \to \mathbb{R}^+ \cup \{+\infty\}$,

$$\chi^1_{x_0}(\varepsilon) = \sup\{\gamma : \exists \delta,\ K,\ \forall \beta \geq 1 - \varepsilon,\ 2^{-j} \leq \delta \Rightarrow |d_{j,[2^j x_0 \pm k_{j,\beta}]}| \leq K 2^{-j\gamma}\}.$$

For $\varepsilon > 0$ small enough, for $\rho \in [0,1]$, $\chi^\rho_{x_0}(\varepsilon)$ relies on the maximum decay rate of some selected wavelet coefficients that lie around x_0.

Definition 2.5. For any $x_0 \in (0,1)$, the 2-microlocal spectrum of f at x_0, $\chi^f_{x_0} : [0,1] \to \mathbb{R}^+ \cup \{+\infty\}$ is the mapping defined by $\chi^f_{x_0}(\rho) = \lim_{\varepsilon \to 0^+} \chi^\rho_{x_0}(\varepsilon)$.

It is proved in [15] that the 2-microlocal spectrum is independent of the choice of the mother wavelet ψ (provided that ψ belongs to the Schwartz class with enough vanishing moments). The quantity $\chi^f_{x_0}(1)$ characterizes the behavior of the wavelet coefficients that lie in the neighborhood of the cone of influence $|k2^{-j} - x_0| \leq 2^{-j}$, while $\chi^f_{x_0}(\rho)$ is related to the behavior of the wavelet coefficients that are located in the space-frequency plane around the curves $|k2^{-j} - x_0| = 2^{-j\rho}$. Eventually, $\chi^f_{x_0}(0)$ characterizes the behavior of the wavelet coefficients lying below all curves $|k2^{-j} - x_0| = 2^{-j\varepsilon}$, $\varepsilon > 0$,

Next proposition, proved in [15], relates the 2-microlocal spectrum $\chi^f_{x_0}$ to the pointwise Hölder exponent $h_f(x_0)$ and the oscillating exponent $\beta_f(x_0)$.

Proposition 2.6. *Let f be a function satisfying (2.2), and assume in addition that $f \in C^\varepsilon([0,1])$ for some $\varepsilon > 0$. Let $x_0 \in (0,1)$. One has*

$$h_f(x_0) = \inf\left\{\chi^f_{x_0}(\rho)/\rho : \rho \in (0,1]\right\}$$

and if $h_f(x_0) < +\infty$,

$$\beta_f(x_0) = \inf\left\{\beta \in [0,+\infty] : \chi^f_{x_0}\left(1/(1+\beta)\right) = h_f(x_0)/(1+\beta)\right\}.$$

In the following, we use this natural notion of 2-microlocal spectrum and the simple characterizations provided by Proposition 2.6 to investigate the local behaviors of continuous functions.

The complementary notion of *upper* 2-microlocal spectrum is also required.

Definition 2.7. Let f be a function satisfying (2.2). For any given $x_0 \in (0,1)$, let

- $\tilde\chi^0_{x_0} : (0,1) \to \mathbb{R}^+ \cup \{+\infty\}$,

$$\tilde\chi^0_{x_0}(\varepsilon) = \inf\{\gamma : \exists \delta, K,\ \forall \beta \leq \varepsilon,\ 2^{-j} \leq \delta \Rightarrow |d_{j,[2^j x_0 \pm k_{j,\beta}]}| \geq K 2^{-j\gamma}\}.$$

- for any $\varepsilon > 0$, $\tilde\chi^\rho_{x_0} : (0, \min(\rho, 1-\rho)) \to \mathbb{R}^+ \cup \{+\infty\}$,

$$\tilde\chi^\rho_{x_0}(\varepsilon) = \inf\{\gamma : \exists \delta, K, \forall \beta \in [\rho - \varepsilon, \rho + \varepsilon],\ 2^{-j} \leq \delta \Rightarrow |d_{j,[2^j x_0 \pm k_{j,\beta}]}| \geq K 2^{-j\gamma}\}.$$

- $\tilde\chi^1_{x_0} : (0,1) \to \mathbb{R}^+ \cup \{+\infty\}$

$$\tilde\chi^1_{x_0}(1-\varepsilon) = \inf\{\gamma : \exists \delta, K,\ \forall \beta \geq 1 - \varepsilon,\ 2^{-j} \leq \delta \Rightarrow |d_{j,[2^j x_0 \pm k_{j,\beta}]}| \geq K 2^{-j\gamma}\}.$$

The upper 2-microlocal spectrum of f at x_0 is the mapping defined by $\tilde{\chi}^f_{x_0} : \rho \in [0,1] \mapsto \lim_{\varepsilon\to 0^+} \tilde{\chi}^\rho_{x_0}(\varepsilon) \in \mathbb{R}^+ \cup \{+\infty\}$.

The mapping $\tilde{\chi}^f_{x_0}$ is related to upper bounds for the decay rate of the wavelet coefficients around x_0. For every x_0, for every $\rho \in [0,1]$, one has $\chi^f_{x_0}(\rho) \leq \tilde{\chi}^f_{x_0}(\rho)$.

3. General results

Recall that the mother wavelet ψ is fixed, and that f is supposed to verify (2.2).

3.1. **An upper bound for the multifractal spectrum.** We recall classical results on the upper bound for the multifractal spectrum of f when $f \in C^\varepsilon([0,1])$. Let us define (3.1)

$$\mathrm{Supp}_d(f) = \left\{ x \in (0,1) : \left\{ \begin{array}{l} \forall\, J \geq 0,\ \forall \varepsilon > 0,\ \exists\, j \geq J,\ \exists k \in \{0,...,2^j-1\} \\ \text{with } |k2^{-j} - x| \leq 2^{-j(1-\varepsilon)},\ \text{such that } d_{j,k} \neq 0 \end{array} \right\} \right\}. \tag{3.1}$$

Heuristically, $\mathrm{Supp}_d(f)$ is the subset of $[0,1]$ constituted by the points x whose cone of influence contains an infinite number of non-zero wavelet coefficients.

Proposition 3.1. 1. *If $0 \leq h \leq \xi'_f(p_c^-)$, then $d_f(h) \leq (\xi_f - 1)^*(h)$.*

2. *If $h \in (\xi'_f(p_c^-), \xi'_f(0^+))$ and $E^f_h \subset \{x \in (0,1) : \min(\underline{\alpha}^-_d(x), \underline{\alpha}_d(x), \underline{\alpha}^+_d(x)) = h\}$, then $d_f(h) \leq (\xi_f - 1)^*(h)$.*

3. *If $h > \xi'_f(0^+)$, then $dim(E^f_h \cap Supp_d(f)) \leq (\xi_f - 1)^*(h)$.*

4. *If $(\xi_f - 1)^*(h) < 0$, then $E^f_h \cap Supp_d(f) = \emptyset$.*

These properties are proved in [12, 13], and in [3] when the wavelet coefficients are deduced from a measure μ. Proposition 3.1 gives a sharp upper bound of the multifractal spectrum of f when $\mathrm{Supp}_d(f) = [0,1]$. But $\mathrm{Supp}_d(f)$ is strictly included in $[0,1]$ for the thresholded wavelet series g^{it} and g^t considered in Section 5, and a finer study is needed on $[0,1]\backslash\mathrm{Supp}_d(f)$.

3.2. **Detection of oscillations.** When the multifractal formalism for functions is not verified by a continuous function f, the presence of oscillating singularities for f is often detected. The following theorem gives a clue on the relation between the failure of the multifractal formalism and the presence of oscillating singularities.

Theorem 3.2. *Let f be a function satisfying (2.2), and assume that $f \in C^\varepsilon([0,1])$ for some $\varepsilon > 0$. Assume that there exists an exponent $h > 0$ such that $\tilde{d}_f(h) < d_f(h)$. Then there exists a set $E \subset [0,1]$ of dimension $d_f(h)$ of oscillating singularities for f such that for every $x \in E$, $h_f(x) = h$.*

The drawback of Theorem 3.2 is that the knowledge of the multifractal spectrum d_f is required to point out oscillating singularities. Nevertheless it enlightens the relations between oscillations and formalisms, see the remarks at the end of this Section.

Proof. Let $\eta > 0$, such that $\tilde{d}_f(h) + \eta < d_f(h)$. By Definition 2.2 of $\tilde{d}_f(h)$, there exist $\varepsilon > 0$ and a scale J such that $j \geq J$ implies

$$j^{-1} \log_2 N^\varepsilon_j(h) \leq \tilde{d}_f(h) + \eta. \tag{3.2}$$

For $j \geq J$, let us denote by $S_{j,\varepsilon}$ the set of wavelet coefficients $d_{j,k}$ such that $2^{-j(h+\varepsilon)} \leq |d_{j,k}| \leq 2^{-j(h-\varepsilon)}$. In view of Definition 2.4, (3.2) is equivalent to say that, at each scale $j \geq J$, the cardinal of $S_{j,\varepsilon}$ is less than $2^{j(\tilde{d}_f(h)+\eta)}$.

Let us estimate the Hausdorff dimension of the sets S_ε^ρ defined for $0<\rho<1$ by

$$S_\varepsilon^\rho = \bigcap_{J\geq 0} T_J \text{ where } T_J = \bigcup_{j\geq J} \bigcup_{k:d_{j,k}\in S_{j,\varepsilon}} \left[k2^{-j}-2^{-j\rho}, k2^{-j}+2^{-j\rho}\right].$$

For every $J\geq 0$, the set S_ε^ρ is obviously covered by the set T_J. Let $d>(\tilde{d}_f(h)+\eta)/\rho$. The cardinal of $S_{j,\varepsilon}$ is upper bounded by $2^{j(\tilde{d}_f(h)+\eta)}$, thus

$$\sum_{k:d_{j,k}\in S_{j,\varepsilon}} \left|\left[k2^{-j}-2^{-j\rho}, k2^{-j}+2^{-j\rho}\right]\right|^d \leq 2^{j(\tilde{d}_f(h)+\eta)}2^{d(1-j\rho)} \leq C2^{j(\tilde{d}_f(h)+\eta-d\rho)},$$

$$\text{and } \sum_{U\in T_J}|U|^d = \sum_{j\geq J}\sum_{k:d_{j,k}\in S_{j,\varepsilon}} \left|\left[k2^{-j}-2^{-j\rho}, k2^{-j}+2^{-j\rho}\right]\right|^d < +\infty.$$

The sum $\sum_{U\in T_J}|U|^d$ goes to zero when $J\to+\infty$. This implies that the d-dimensional Hausdorff measure of S_ε^ρ equals zero, for any $d>(\tilde{d}_f(h)+\eta)/\rho$. As a consequence, the Hausdorff dimension of S_ε^ρ is smaller than $(\tilde{d}_f(h)+\eta)/\rho$.

Let us study the complementary set of S_ε^ρ, that we denote by C_ε^ρ. If $x\in C_\varepsilon^\rho$, there exists a scale J_x such that for every $j\geq J_x$, for every k such that $|k2^{-j}-x|\leq 2^{-j\rho}$, one has $|d_{j,k}|\geq 2^{-j(h-\varepsilon)}$ or $|d_{j,k}|\leq 2^{-j(h+\varepsilon)}$. This means that for every $\rho'\in(\rho,1]$, $\chi_x^f(\rho')\in(0,h-\varepsilon)\cup(h+\varepsilon,+\infty]$. In particular, if $h_f(x)=h$, applying Proposition 2.6 gives that $\beta_f(x)>0$, i.e. there are fast oscillations around x.

Now, let us take $\rho<1$ such that $(\tilde{d}_f(h)+\eta)/\rho<d_f(h)$. One has

$$E_h^f = \left(E_h^f\cap S_\varepsilon^\rho\right)\cup\left(E_h^f\cap C_\varepsilon^\rho\right)$$

Let $E=E_h^f\cap C_\varepsilon^\rho$. Since $\dim_H E_h^f\cap S_\varepsilon^\rho \leq (\tilde{d}_f(h)+\eta)/\rho < d_f(h)$, the set E has a Hausdorff dimension equal to $d_f(h)$. For every $x\in E$, $x\in E_h^f$ thus $h_f(x)=h$, and at the same time, since $E\subset C_\varepsilon^\rho$, $\beta_f(x)\geq 1/\rho-1>0$, i.e. x is an oscillating singularity for f. □

The condition $\tilde{d}_f(h)<d_f(h)$ allows to find an upper bound for the dimensions of the sets S_ε^ρ. Unfortunately, it is impossible to treat in general the case $\tilde{d}_f(h)>d_f(h)$, since this inequality does not yield neither a sharp upper bound nor a lower bound for $\dim_H S_\varepsilon^\rho$.

If the Legendre transform of $\xi-1$ at exponent h is strictly less than $d_f(h)$, i.e. if the multifractal formalism (as stated in Definition 2.3) fails at h, then $\tilde{d}_f(h)<d_f(h)$ and Theorem 3.2 can be applied. Thus Theorem 3.2 emphasizes that if the multifractal formalism is not satisfied at exponent h, then somehow there are not enough wavelet coefficients to have for every $x\in E_h^f$, $\chi_x^f(1)=h$ (or equivalently $\beta_f(x)=0$). Hence there are oscillating singularities in E_h^f. Remark that the property $(\xi-1)^*(h)<d_f(h)$ can only be satisfied when $h\geq(\xi-1)'(p_c)$, because of the upper bound (2.3) proved in [12]. This is the reason why, as noticed in [2], working with the large deviation spectrum yields more informations than working on the Legendre spectrum. In particular, the conditions of Theorem 3.2 can occur for every $h\geq 0$.

Theorem 3.2 also gives an answer to a question addressed by S. Jaffard and his collaborators in [12, 16, 8]. In these articles, they investigate the typical local behaviors of functions in some functional spaces V. They prove that respectively "quasi-all" [12] and "almost every" function [8] are multifractal functions, with the same multifractal spectrum depending of course on V. "Quasi-all" functions means every function in an intersection of dense open sets, while "almost every" function refers to the notion of prevalence in metric spaces. As a by-product of their work, they obtain that "quasi-all" and "almost every" functions do not obey our multifractal formalism for functions. An interesting and natural problem is the seek for oscillating singularities for these typical multifractal functions. Indeed, as mentioned before, the failure of the multifractal formalism is often detected together with the presence of oscillating singularities. In particular, in [16] it is proved that "quasi-all" functions (thus in the sense of Baire's categories) have oscillating singularities. The question is also to be addressed in the prevalence case.

Actually, the answer to two these questions is provided by Theorem 3.2. Indeed, as a direct consequence of Theorem 3.2, since "quasi-all" and "almost every" functions in these functional spaces V mentioned above do not verify our multifractal formalism for functions (this especially holds true for the so-called "saturating functions"), these functions also have dense sets of oscillating singularities.

3.3. **Creating oscillations.** Let us introduce the wavelet threshold operators.

Definition 3.3. Let f be a function satisfying (2.2). Let $\gamma > 0$. The function series f^t, defined by $f^t = \sum_{j\geq 1}\sum_{k=0}^{2^j-1} d^t_{j,k}\psi_{j,k}$, where

$$d^t_{j,k} = d_{j,k}1\!\!1_{|d_{j,k}|\geq 2^{-j\gamma}},$$

is said to be obtained from f after *a threshold of order* γ. This function series f^t also satisfies (2.2).

We learn from Theorem 3.2 that for any continuous enough function f, $\tilde{d}_f(h) < d_f(h)$ for some exponent $h > 0$ ensures the existence of oscillating singularities for f. For such a function f, if $d_f(h) > 0$ for some h 0, a threshold of order $\gamma < h$ imposes $\tilde{d}_{f^t}(h) = 0$. But since a threshold increases (local and global) regularity, every point $x \in E^h_f$ has a pointwise Hölder exponent for f^t at x which is greater than h. These points are good candidates to be oscillating singularities for f^t.

Theorem 3.4. *Let f be a function satisfying (2.2), and assume that $f \in C^\varepsilon([0,1])$ for some $\varepsilon > 0$. Assume that $E^h_f \neq \emptyset$. Let f^t be the function obtained after a threshold of f of order $\gamma < h$. Then for every $x \in E^h_f$, either $h_{f^t}(x) = +\infty$, or x is an oscillating singularity for f^t.*

Proof. Since we cancel wavelet coefficients, the threshold we apply obviously increases the local regularity at each point. Thus $h_{f^t}(x) \geq h_f(x)$ for every x.

Let $x \in E^f_h$. We denote by χ^f_x the 2-microlocal spectrum of f at x, and by $\chi^{f^t}_x$ the 2-microlocal spectrum of f^t. One has $h_f(x) = \inf_{\rho\in(0,1]}\chi^f_x(\rho)/\rho = h$, and

$$h_{f^t}(x) = \inf_{\rho\in(0,1]}\chi^{f^t}_x(\rho)/\rho \geq h. \tag{3.3}$$

For every (j,k), one has either $|d_{j,k}| \geq 2^{-j\gamma}$ or $d_{j,k} = 0$. This implies that

$$\forall\rho\in[0,1],\ \chi^{f^t}_x(\rho) \in [\varepsilon,\gamma]\cup\{+\infty\}. \tag{3.4}$$

Assume $h_{f^t}(x) < +\infty$. Combining (3.3) and (3.4) implies that there exists $\rho_x \in (0,1)$ such that $\chi^{f^t}_x(\rho_x)/\rho_x = h_{f^t}(x)$.
Let $\rho_h < 1$ be the unique real number such that $\gamma/\rho_h = h$. By construction, for every $\rho \in (\rho_h, 1]$, $\chi^{f^t}_x(\rho)/\rho \in [\varepsilon/\rho, \gamma/\rho]\cup\{+\infty\}$. In particular, for every $\rho\in(\rho_h,1]$, $\chi^{f^t}_x(\rho)/\rho \neq h_{f^t}(x)$.

In view of Proposition 2.6, this exactly means that $\beta_{f^t}(x) \in [1/\rho_h - 1, 1/\rho_x - 1]$, i.e. that $\beta_{f^t}(x) > 0$ and f^t has an oscillating singularity at x. □

Theorem 3.4 shall be thought as a sort of Gibbs phenomenon that occurs when forgetting at each scale j in the reconstruction formula (2.1) of the initial function f the smallest wavelet coefficients.

The existence of oscillating singularities depends on the function f and on the choice of the parameter γ. Nevertheless, in the setting of Theorem 3.4, for γ close enough to h, a reasonable hope is that there remains some points x with $h_{f^t}(x) < +\infty$. This is the case with the wavelet series g we focus on in the following sections: The wavelet series g^t obtained after thresholding has dense sets of oscillating singularities. In addition the multifractal formalism fails for g^t.

In signal and image processing, the *hard* threshold [6, 7] is a powerful method used to compress data. It consists in putting to zero all the wavelet coefficients $d_{j,k}$ that are too small, i.e. such that $|d_{j,k}| \leq \varepsilon$ for a given constant $\varepsilon > 0$. The hard threshold can certainly be compared to the threshold we define in Definition 3.3 in the following way. Let f be a discrete signal of size $N = 2^J$, and consider its wavelet decomposition $f = f_0 + \sum_{j=0}^{J-1} \sum_{k=0}^{2^j} d_{j,k}\psi_{j,k}$ (where f_0 corresponds to the low-frequency part of f). For ε small enough, applying a hard threshold amounts to putting to 0, at each scale j, the wavelet coefficients $d_{j,k}$ such that $|d_{j,k}| \leq 2^{-j\frac{-\log_2 \varepsilon}{j}}$. If one considers only the smallest scales (for example the scales j such that $j \geq [N/2]$), the hard threshold puts to 0 more coefficients than a threshold of order $\gamma^t = (2\log_2 \varepsilon)/J$. This means that the points with (discrete) local regularity greater than γ^t become either C^∞ points, or oscillating singularities. Of course real data are discrete and the number of available scales is finite, thus every point has an infinite Hölder exponent. Nevertheless, this phenomenon we observe at small scales has repercussions on the appearance of the signal and/or image at coarse scales, and is worth to be further studied.

We state, without proof, the analog of Theorem 3.4 for the inverse threshold of order γ.

Theorem 3.5. *Let f be a function satisfying (2.2), and assume that $f \in C^\varepsilon([0,1])$ for some $\varepsilon > 0$. Assume that $E_f^h \neq \emptyset$. Let f^{it} be the function obtained after an inverse threshold of f of order $\gamma < h$. Then for every $x \in E_f^h$, either $h_{f^{it}}(x) = +\infty$, or x is an oscillating singularity for f^{it}.*

4. Study of a specific example

Let us first begin with a recall on local regularity of measures.

4.1. Local regularity of measures.

If $x \in (0,1)$, $\forall j \geq 1$, $k_{j,x}$ is the unique integer such that $x \in [k_{j,x}2^{-j}, (k_{j,x}+1)2^{-j})$, and $k_{j,x}^+ = k_{j,x} + 1$, $k_{j,x}^- = k_{j,x} - 1$. By convention, $\log(0) = -\infty$.

Let μ be a positive Borel measure on $[0,1]$. For $x \in (0,1)$, the lower and upper Hölder exponent of μ at x are respectively defined by

$$\underline{\alpha}_\mu(x) = \liminf_{j\to+\infty} \frac{\log \mu(I_{j,k_{j,x}})}{\log |I_{j,k_{j,x}})|} \quad \text{and} \quad \overline{\alpha}_\mu(x) = \limsup_{j\to+\infty} \frac{\log \mu(I_{j,k_{j,x}})}{\log |I_{j,k_{j,x}}|}$$

When $\underline{\alpha}_\mu(x) = \overline{\alpha}_\mu(x)$, their common value is denoted $\alpha_\mu(x)$ and called the Hölder exponent of μ at x. Similarly, the left and right lower and upper Hölder exponents of μ at x are defined by

$$\underline{\alpha}_\mu^-(x) = \liminf_{j\to+\infty} \frac{\log \mu(I_{j,k_{j,x}^-})}{\log |I_{j,k_{j,x}^-}|} \quad \text{and} \quad \underline{\alpha}_\mu^+(x) = \liminf_{j\to+\infty} \frac{\log \mu(I_{j,k_{j,x}^+})}{\log |I_{j,k_{j,x}^+}|},$$

$$\overline{\alpha}_\mu^-(x) = \limsup_{j\to+\infty} \frac{\log \mu(I_{j,k_{j,x}^-})}{\log |I_{j,k_{j,x}^-})|} \quad \text{and} \quad \overline{\alpha}_\mu^+(x) = \limsup_{j\to+\infty} \frac{\log \mu(I_{j,k_{j,x}^+})}{\log |I_{j,k_{j,x}^+}|}.$$

When they coincide (i.e. when the limits exist), the left and right Hölder exponents of μ at x are defined by $\alpha_\mu^-(x) = \underline{\alpha}_\mu^-(x) = \overline{\alpha}_\mu^-(x)$ and $\alpha_\mu^+(x) = \underline{\alpha}_\mu^+(x) = \overline{\alpha}_\mu^+(x)$

The following level sets for μ are also considered.

Definition 4.1. For every $\alpha \geq 0$, define

$$\begin{aligned}
E_\alpha^\mu &= \{x \in (0,1) \cap \text{Supp}(\mu) : \alpha_\mu(x) = \alpha\}, \\
F_\alpha^\mu &= \left\{x \in (0,1) \cap \text{Supp}(\mu) : \min(\underline{\alpha}_\mu(x), \underline{\alpha}_\mu^+(x), \underline{\alpha}_\mu(x)^-) = \alpha\right\}, \\
G_\alpha^\mu &= \left\{x \in (0,1) \cap \text{Supp}(\mu) : \alpha_\mu(x) = \alpha_\mu^+(x) = \alpha_\mu^-(x) = \alpha\right\}, \\
H_\alpha^\mu &= \left\{x \in (0,1) \cap \text{Supp}(\mu) : \max(\overline{\alpha}_\mu(x), \overline{\alpha}_\mu^+(x), \overline{\alpha}_\mu(x)^-) = \alpha\right\}.
\end{aligned}$$

One obviously has $G_\alpha^\mu \subset E_\alpha^\mu \cap F_\alpha^\mu \cap H_\alpha^\mu$.

The *multifractal spectrum* of μ is the mapping $\alpha \geq 0 \to d_\mu(\alpha) = \dim_H E_\alpha^\mu$. It measures the size of the level sets of the Hölder exponent.

4.2. **Definition of the wavelet series g associated with a positive Borel measure μ.** The binomial measure μ associated with the couple (q_0, q_1) is built as follows. Starting from the uniform measure μ^0 on $I_0 = [0,1]$, one splits I_0 into the two dyadic intervals $I_{1,0} = [0,1/2)$ and $I_{1,1} = [1/2,1]$, and defines a measure μ^1 by $\mu^1(I_{1,0}) = q_0$ and $\mu^1(I_{1,1}) = q_1$ (uniformly in each interval). Iterating this scheme, one obtains, at each scale $j \geq 1$, a measure μ^j whose distribution is constant over each dyadic interval $I_{j,k} = [k2^{-j}, (k+1)2^{-j})$, and which associates with $I_{j,k}$ the uniform mass $q_0^{\phi(j,k)} q_1^{j-\phi(j,k)}$, where $\phi(j,k)$ is the number of 0 among the j first coordinates in the dyadic decomposition of $k2^{-j}$. The sequence of measures $\{\mu^j\}_{j\geq 1}$ converges weakly to a probability measure μ, called the binomial measure of ratios (q_0, q_1).

The measure μ is known to satisfy the multifractal formalism as defined in [5], and its multifractal spectrum is given by the Legendre transform (1.3) of the scaling function $\tau(q)$ (given by (1.4)). More precisely,

$$\begin{aligned}
&\text{if } \inf_{q\in\mathbb{R}}(q\alpha - \tau(q)) \geq 0, \qquad d_\mu(\alpha) = \inf_{q\in\mathbb{R}}(q\alpha - \tau(q)), \\
&\text{if } \inf_{q\in\mathbb{R}}(q\alpha - \tau(q)) < 0, \qquad d_\mu(\alpha) = -\infty \text{ and } E_\alpha^\mu = \emptyset.
\end{aligned}$$

We denote by Supp (d_μ) the support of d_μ, i.e. Supp $(d_\mu) = \{\alpha \geq 0 : d_\mu(\alpha) \geq 0\}$.

Starting from μ (actually it can be done with any positive Borel measure), one builds using the same method as in [3] a wavelet series g on $\mathbb{R}$ with the formula (1.5)

$$g(x) = \sum_{j\geq 1} \sum_{k=0,\dots,2^j-1} d_{j,k}\psi_{j,k}(x) \text{ with } d_{j,k} = 2^{-j(s_0-1/p_0)}(\mu(I_{j,k}))^{1/p_0}, \ \forall j, \ \forall k.$$

The next lemma [5, 3] is useful for the multifractal analysis of the series g.

Lemma 4.2. *Let μ be the binomial measure with ratios (q_0, q_1). For every $\alpha \in$ Supp (d_μ), one has* $\dim_H(E_\alpha^\mu) = \dim_H(F_\alpha^\mu) = \dim_H(G_\alpha^\mu) = \dim_H(H_\alpha^\mu) = \tau^*(\alpha)$.

4.3. **Regularity at $x \in G^\mu_\alpha$.** In the following, $d_{j,k} = 2^{-j(s_0-1/p_0)}(\mu(I_{j,k}))^{1/p_0}$ denote the wavelet coefficients of the wavelet series g.

Proposition 4.3. *For every $\alpha \in$ Supp (d_μ), if $x \in G^\mu_\alpha$, then $\forall \rho \in [0,1]$,*

$$\chi^g_x(\rho) = s_0 - 1/p_0 + (\rho\alpha - (1-\rho)\log_2 q_1)/p_0, \tag{4.1}$$

$$\tilde{\chi}^g_x(\rho) = s_0 - 1/p_0 + (\rho\alpha - (1-\rho)\log_2 q_0)/p_0. \tag{4.2}$$

Proof. For all $\rho \in (0,1)$, define the sets of wavelet coefficients $B_{\rho,j}$ and B^J_ρ by $B_{\rho,j} = \{d_{j,k} : |k2^{-j} - x| \le 2^{-j\rho}\}$, and $B^J_\rho = \cup_{j \ge J} B_{j,\rho}$ (they depend on x).

Remark that if there exist a constant C and an integer J such that, for every coefficient $d_{j,k} \in B^J_\rho$,

$$C^{-1}2^{-j\gamma_1} \le |d_{j,k}| \le C2^{-j\gamma_2}$$

(where $\gamma_1,\ \gamma_2 > 0$), then $\forall \rho' > \rho$, $\gamma_1 \le \chi^g_x(\rho') \le \tilde{\chi}^g_x(\rho') \le \gamma_2$. Indeed, remembering Definitions 2.4 to 2.7, the computation of $\chi^g_x(\rho')$ and $\tilde{\chi}^g_x(\rho')$ takes into account only wavelet coefficients that lie in B^J_ρ.

Fix $\varepsilon > 0$. $x \in G^\mu_\alpha$ implies that there exists $J > 0$ such that $j \ge J$ implies

$$2^{-j(\alpha+\varepsilon)} \le \log \mu(I_{j,k^*_{j,x}}) \le 2^{-j(\alpha-\varepsilon)}, \tag{4.3}$$

for every $k^*_{j,x} \in \{k^-_{j,x}, k_{j,x}, k^+_{j,x}\}$. Fix $\rho \in (0,1)$, and $\rho' \in (\rho, 1]$. Let $j \ge (J+1)/\rho$, $k \in \{0, \dots, 2^j - 1\}$, such that $d_{j,k} \in B^j_{\rho'}$. One has

$$I_{j,k} \subset [x - 2^{-j\rho}, x + 2^{-j\rho}] \subset I_{j_\rho, k_{j_\rho,x}-1} \cup I_{j_\rho, k_{j_\rho,x}} \cup I_{j_\rho, k_{j_\rho,x}+1},$$

where $j_\rho = [j\rho] \ge J$. Thus $\mu(I_{j,k}) \le \max\big(\mu(I_{j_\rho,k_{j_\rho,x}-1}), \mu(I_{j_\rho,k_{j_\rho,x}}), \mu(I_{j_\rho,k_{j_\rho,x}+1})\big)$. But if $I_{j'',k''}$ is a subinterval of $I_{j',k'}$ with $|I_{j'',k''}| = 1/2|I_{j',k'}|$, one has either $\mu(I_{j'',k''}) = q_0\mu(I_{j',k'})$ or $\mu(I_{j'',k''}) = q_1\mu(I_{j',k'})$. Thus since $I_{j,k}$ is a subinterval of one of the intervals $\{I_{j_\rho,k_{j_\rho,x}-1}, I_{j_\rho,k_{j_\rho,x}}, I_{j_\rho,k_{j_\rho,x}+1}\}$, one obtains

$$\begin{aligned} \mu(I_{j,k}) &\le \min\big(\mu(I_{j_\rho,k_{j_\rho,x}-1}), \mu(I_{j_\rho,k_{j_\rho,x}}), \mu(I_{j_\rho,k_{j_\rho,x}+1})\big) q_1^{j-j_\rho+1} \\ \text{and } \mu(I_{j,k}) &\ge \max\big(\mu(I_{j_\rho,k_{j_\rho,x}-1}), \mu(I_{j_\rho,k_{j_\rho,x}}), \mu(I_{j_\rho,k_{j_\rho,x}+1})\big) q_0^{j-j_\rho+1}. \end{aligned}$$

Combining this with (4.3) yields $q_0^{j-j_\rho+1}2^{-j_\rho(\alpha+\varepsilon)} \le \mu(I_{j,k}) \le q_1^{j-j_\rho+1}2^{-j_\rho(\alpha-\varepsilon)}$, or equivalently since $j_\rho \sim [j\rho]$

$$C^{-1}2^{-j(\rho(\alpha+\varepsilon)-(1-\rho)\log_2 q_0)} \le \mu(I_{j,k}) \le C2^{-j(\rho(\alpha-\varepsilon)-(1-\rho)\log_2 q_1)}.$$

Let us denote $\phi_0(\rho) = s_0 - 1/p_0 + (\rho\alpha - (1-\rho)\log_2 q_0)/p_0$ and $\phi_1(\rho) = s_0 - 1/p_0 + (\rho\alpha - (1-\rho)\log_2 q_1)/p_0$. Coming back to the wavelet coefficients, one writes that for any $d_{j,k} \in B_{j,\rho'}$, $C^{-1}2^{-j(\phi_0(\rho)+\rho\varepsilon/p_0)} \le d_{j,k} \le C2^{-j(\phi_1(\rho)-\rho\varepsilon/p_0)}$. This holds true for any $j \ge J/\rho$, thus also for any wavelet coefficients $d_{j,k} \in B^{J/\rho}_{\rho'}$. Applying the first remark leads to

$$\phi_1(\rho) - \rho\varepsilon/p_0 \le \chi^g_x(\rho') \text{ and } \tilde{\chi}^g_x(\rho') \le \phi_0(\rho) + \rho\varepsilon/p_0.$$

This is true for any $\rho' \in (\rho, 1]$, but stays also true for any $\rho \in (0,1]$. Hence $\phi_1(\rho) - \rho\varepsilon/p_0 \le \chi^g_x(\rho)$ and $\tilde{\chi}^g_x(\rho) \le \phi_0(\rho) + \rho\varepsilon/p_0$. Letting ε go to zero gives

$$\phi_1(\rho) \le \chi^g_x(\rho) \text{ and } \tilde{\chi}^g_x(\rho) \le \phi_0(\rho). \tag{4.4}$$

The optimality of (4.4) is a direct consequence of the following argument.
If ρ and j are fixed, $I_{j_\rho,k_{j_\rho,x}}$ is strictly included in $[x - 2^{-j\rho}, x + 2^{-j\rho}]$. The same holds

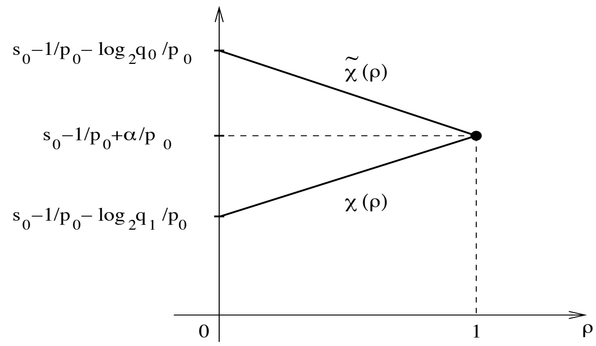

FIGURE 2. Plot of the 2-microlocal spectrum and upper 2-microlocal spectrum when $x \in G_\alpha^\mu$.

for all the dyadic subintervals of $I_{j_\rho,k_{j_\rho,x}}$. In particular, there exist two dyadic intervals I_{j,k_1} and I_{j,k_2}, subintervals of $I_{j_\rho,k_{j_\rho,x}}$, such that

$$\mu(I_{j,k_1}) = q_0^{j-j_\rho+1}\mu(I_{j_\rho,k_{j_\rho,x}}) \text{ and } \mu(I_{j,k_2}) = q_1^{j-j_\rho+1}\mu(I_{j_\rho,k_{j_\rho,x}}).$$

This translates to the wavelet coefficients d_{j,k_2} into

$$|d_{j,k_2}| \geq C2^{-j(\phi_1(\rho)+\rho\varepsilon/p_0)}. \tag{4.5}$$

Such coefficients exist in $B_{j,\rho}$ at every scale $j \geq J\rho$. Thus, there exists $\rho_\varepsilon \in [\rho, 1]$ such that $\chi_x^g(\rho_\varepsilon) \leq s\phi_1(\rho) + \rho\varepsilon/p_0$. Combining this with (4.4) yields that ρ_ε is close to ρ. In particular, letting ε go to zero, one gets $\chi_x^g(\rho) \leq \phi_1(\rho)$, and (4.1) is proved.

The optimality of the second inequality of (4.4) is proved by applying the same study for the coefficients d_{j,k_1}, and (4.2) is obtained. Remark that $\chi_x^g(1) = \tilde\chi_x^g(1) = s_0 - 1/p_0 + \alpha/p_0 = h$.

Eventually, it is obvious that for all (j,k), $d_{j,k} \leq 2^{-j(s_0-1/p_0+-\log_2 q_1/p_0)}$, thus $\chi_x^g(0) = s_0 - 1/p_0 + -\log_2 q_1/p_0$. The lower bound $d_{j,k} \geq 2^{-j(s_0-1/p_0+-\log_2 q_0/p_0)}$ yields by the same argument $\tilde\chi_x(0) = s_0 - 1/p_0 + -\log_2 q_0/p_0$. □

Remark: (4.1) remains true if $x \in F_\alpha^\mu$, i.e. if the values of the lower Hölder exponents of μ at x are controlled. Indeed, for such a point x, a slight modification of the above proof shows that there still exists a infinite number of j's such that (4.3), (4.5) and then (4.1) hold. Similarly, the value of the upper 2-microlocal spectrum depends only on the upper Hölder exponents of μ at x.

Proposition 4.4. *For every $x \in (0,1)$, there exist two real numbers $0 < \alpha \leq \beta$ such that $x \in F_\alpha^\mu$ and $x \in H_\beta^\mu$. Then, for every $\rho \in [0,1]$,*

$$\chi_x^g(\rho) = s_0 - 1/p_0 + (\rho\alpha - (1-\rho)\log_2 q_1)/p_0, \tag{4.6}$$

$$\tilde\chi_x^g(\rho) = s_0 - 1/p_0 + (\rho\beta - (1-\rho)\log_2 q_0)/p_0. \tag{4.7}$$

In particular, there is no fast oscillation around x, i.e. $\beta_g(x) = 0$, and

$$h_g(x) = s_0 - 1/p_0 + \alpha/p_0. \tag{4.8}$$

Proof. It remains to prove (4.8). Proposition 2.6 yields that $h_g(x) = \inf_\rho \{\chi_x^g(\rho)/\rho\} = s_0 - 1/p_0 + \alpha/p_0 = h$, and, since $\chi_x^g(1) = h_g(x)$, $\beta_g(x) = 0$. □

Remarks: 1. As proved in [3], (4.8) remains true for any positive Borel measure μ, and not only for the binomial measure. Nevertheless, (4.6) and (4.7) are not true in general and require additional properties on μ.
2. Fix x and $\rho \in (0,1]$. It is easy too see that for every $\varepsilon > 0$, for every $\gamma \in [\chi_x^g(\rho), \tilde{\chi}_x^g(\rho)]$, there exists an infinite number of scale j such that

$$\left| -j^{-1} \log_2 |d_{j,k}| - \gamma \right| \leq \varepsilon. \tag{4.9}$$

It is now easy to compute the multifractal spectrum of g.

Proposition 4.5. *The wavelet series g obeys the multifractal formalism for functions as defined in Definition 2.3, and $d_g(h) = \dim(E_h^g \cap [0,1]) = d_\mu\big(p_0(h - (s_0 - 1/p_0))\big)$.*

Proof. From Proposition 4.3, one gets that for every $x \in G_\alpha^\mu$, $h_g(x) = h = s_0 - 1/p_0 + \alpha/p_0$, and from Proposition 4.2, one deduces that $d_g(h) \geq \dim_H(G_\alpha^\mu) = \tau^*(\alpha)$. An easy computation shows that

$$\xi_g(p) = 1 + s_0 - 1/p_0 + \tau(p/p_0). \tag{4.10}$$

Applying Proposition 3.1 for g gives $d_g(h) \leq (\xi_g - 1)^*(h) = \tau^*(\alpha)$, and thus $d_g(h) = d_{\mu_0}(\alpha)$ (remark that for g, $\text{Supp}_d(f)$ as defined in (3.1) equals $[0,1]$). □

Remark: Proposition 4.5 is a simple application of the main Theorem of [3], where it is shown that, provided that a positive Borel measure ν on $[0,1]$ satisfies a certain multifractal formalism for measures (this holds true for the binomial measure μ), the corresponding wavelet series g_ν built as in (1.5) satisfies the multifractal formalism for functions we defined.

5. Multifractal spectrum of g^{it} and g^t

The function series g has only cusps (i.e. non-oscillating singularities). Theorem 3.4 proposes a simple method to create oscillating singularities. Our purpose here is to apply Theorem 3.4 to the wavelet series g. We see that oscillating singularities are effectively created, and that non-concave multifractal spectra may be obtained. We finish the paper by a remark on the non-verification of the multifractal spectrum by the truncated versions of g.

Before testing this method on g, we begin by the inverse threshold of order γ we mentioned in Section 3.

We set $h_{\min} = s_0 - 1/p_0 + \alpha_{\min}/p_0$, $h_s = s_0 - 1/p_0 + \alpha_s/p_0$ and $h_{\max} = s_0 - 1/p_0 + \alpha_{\max}/p_0$.

5.1. Hausdorff multifractal spectrum of g^{it}.

Let $\gamma \in [h_{\min}, h_{\max}]$. Recall that the truncated wavelet series g^{it} is defined as $g^{it} = \sum_{j,k} d_{j,k}^{it} \psi_{j,k}$, where

$$d_{j,k}^{it} = d_{j,k} \mathbf{1}_{|d_{j,k}| \leq 2^{-j\gamma}}. \tag{5.1}$$

Theorem 5.1. *Let $\omega_{it} : [h_{\min}, \gamma] \to (0, +\infty)$ be the decreasing function defined by*

$$u \to \gamma \frac{u - (s_0 - 1/p_0) + \log_2 q_0/p_0}{\gamma - (s_0 - 1/p_0) + \log_2 q_0/p_0} \tag{5.2}$$

The multifractal spectrum of f^{it} ranges in $[\gamma, \max(h_{\max}, \omega_{it}^{-1}(h_{\min}))]$, and equals

$$d_{f^{it}}(h) = \max(d_f(h), d_f(\omega_{it}^{-1}(h))). \tag{5.3}$$

The mapping ω_{it}^{-1} is the inverse function of ω_{it}: for $h \in (0, +\infty)$, $\omega_{it}^{-1}(h)$ is the unique real number such that $\omega_{it}(\omega_{it}^{-1}(h)) = h$.

Proof. For every (j,k), $|d_{j,k}| \leq 2^{-j\gamma}$, thus $d_{f^{it}}(h) = -\infty$ if $h < \gamma$. We simply denote by χ_x^{it} and $\tilde{\chi}_x^{it}$ the 2-microlocal spectra of g^{it} at x.

Let α and β be two real numbers such that $\alpha_{\min} \leq \alpha \leq \beta \leq \alpha_{\max}$. Then $E_{\alpha,\beta}$ is the set defined by

$$E_{\alpha,\beta} = \{x : x \in F_\alpha^\mu \text{ and } x \in H_\beta^\mu\}.$$

From [19] one gets that $\dim_H(E_{\alpha,\beta}) = \min(\tau^*(\alpha), \tau^*(\beta))$ for the binomial measure μ. Moreover, for a fixed $\beta > 0$, $\cup_{\alpha \leq \beta} E_{\alpha,\beta} = H_\beta^\mu$, thus $\dim_H(\cup_{\alpha\leq\beta} E_{\alpha,\beta}) = \tau^*(\beta)$. Similarly, for a fixed α, $\cup_{\alpha\leq\beta} E_{\alpha,\beta} = F_\alpha^\mu$, and $\dim_H(\cup_{\alpha\leq\beta} E_{\alpha,\beta}) = \tau^*(\alpha)$.

Let $x \in [0,1]$ such that $x \in E_{\alpha,\beta}$ for some α, $\beta > 0$. Proposition 4.4 and equations (4.6) and (4.7) give us the values of the 2-microlocal spectra of g at x. Property (4.9) is here essential: if for a given x and a given ρ, $\chi_x^g(\rho) \leq \gamma \leq \tilde{\chi}_x^g(\rho)$, then, after applying (5.1), one gets $\chi_x^{it}(\rho) = \gamma \leq \tilde{\chi}_x^{it}(\rho)$ for g^{it}. Let us distinguish three cases.

- if $\gamma \leq s_0 - 1/p_0 + \alpha/p_0$**:** the 2-microlocal spectrum of g^{it} at x becomes

$$\chi_x^{it}(\rho) = \begin{cases} \gamma & \text{if } \rho \in [0,\rho_x] \\ s_0 - 1/p_0 + (\rho\alpha - (1-\rho)\log_2 q_1)/p_0 & \text{if } \rho \in (\rho_x, 1] \end{cases},$$

where ρ_x is the unique ρ such that $s_0 - 1/p_0 + (\rho_x\alpha - (1-\rho_x)\log_2 q_1)/p_0 = \gamma$. Thus $h_{g^{it}}(x) = h_g(x) = \chi_x^{it}(1) = s_0 - 1/p_0 + \alpha/p_0$. The upper 2-microlocal spectrum at x remains unchanged after the application of the threshold of order γ..

- $s_0 - 1/p_0 + \alpha/p_0 \leq \gamma \leq s_0 - 1/p_0 + \beta/p_0$**:** the 2-microlocal spectrum of g^{it} at x becomes constant and equals γ for every $\rho \in [0,1]$. Hence $h_{g^{it}}(x) = \chi_x^{it}(1) = \gamma > s_0 - 1/p_0 + \alpha = h_g(x)$. The upper 2-microlocal spectrum at x remains unchanged.

- $s_0 - 1/p_0 + \beta/p_0 < \gamma$**:** the 2-microlocal spectrum of g^{it} at x is

$$\chi_x^{it}(\rho) = \begin{cases} \gamma & \text{if } \rho \in [0,\rho_x] \\ +\infty & \text{if } \rho \in (\rho_x, 1] \end{cases},$$

where ρ_x is the unique ρ such that $s_0 - 1/p_0 + (\rho_x\beta - (1-\rho_x)\log_2 q_0)/p_0 = \gamma$. Thus $h_{g^{it}}(x) = \inf_\rho \chi_x^{it}(\rho)/\rho = \gamma/\rho_x$. Remark that ρ_x is explicitly given by

$$\rho_x = \frac{\gamma - (s_0 - 1/p_0) + \log_2 q_0/p_0}{p_0(\beta + \log_2 q_0)},$$

and thus $h_{g^{it}}(x) = \omega_{it}^{-1}(s_0 - 1/p_0 + \beta/p_0)$, where ω_{it} is the continuous injective function defined in (5.2). In particular, since $x \notin \text{Supp}(_d(f))$, Proposition 2.6 gives that x is an oscillating singularity for g^{it} with an oscillating exponent equal to $\beta_g(x) = (1+\rho_x)^{-1}$.

Let us finally compute the multifractal spectrum of g^{it}.

The range of ρ's when $x \in [0,1]$ is $\left[\frac{\gamma - (s_0 - 1/p_0) + \log_2 q_0/p_0}{(\log_2 q_0 - \log_2 q_1)/p_0}, 1\right] = [\rho_{\min}, 1]$ (it is minimal for $\beta = \alpha = -\log_2 q_1$, i.e. for $x \in G^\mu_{-\log_2 q_1}$). We denote by $h_{\max}^{it}$ the maximal value of the pointwise Hölder exponent $h_{\max}^{it} = \omega_{it}\big(s_0 - 1/p_0 - \log_2 q_1/p_0\big) = \gamma/\rho_{\min}$.

We recall that $E_h^{g^{it}}$ is the set of points $x \in [0,1]$ such that $h_{g^{it}}(x) = h$. First note that if $x \in E_\gamma^{g^{it}}$, from the above computations one deduces that $\alpha = \gamma$ or $\beta = \gamma$. Thus $d_{g^{it}}(\gamma) = d_g(\gamma) = \tau^*(\gamma)$. Two cases are distinguished:

- $h_{\max}^{it} \leq h_{\max}$**:** If $x \in E_h^{fg^{it}}$ with $\gamma < h \leq h_{\max}^{it}$, x either belongs to a set $E_{\alpha,\beta'}$ with $\beta' \geq \alpha$ and $s_0 + 1/p_0 + \alpha/p_0 = h$, or to a set $E_{\alpha',\beta}$ where $\beta = \omega_{it}^{-1}(h) < p_0(\gamma - (s_0 + 1/p_0))$ and $\alpha' \leq \beta$. Reciprocally,

$$E_h^{g^{it}} \subset (\cup_{\beta'\geq\alpha} E_{\alpha,\beta'}) \cup \big(\cup_{\alpha' \leq \omega_{it}^{-1}(h)} E_{\alpha',\omega_{it}^{-1}(h)}\big).$$

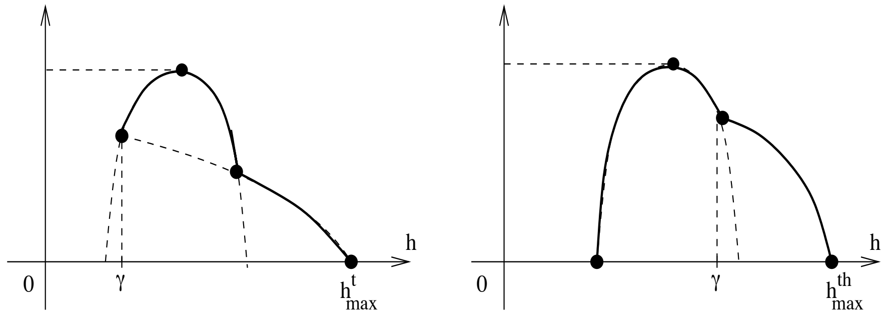

FIGURE 3. Multifractal spectrum of f^{it} (left) and f^t (right) when $h_{\min} < \gamma < h_{\max}$.

Using the remark of the beginning of this proof, one has $\dim_H E_h^{g^{it}} = d_{g^{it}}(h) = \max(\tau^*(\alpha), \tau^*(\omega_{it}^{-1}(h)))$.

Eventually, if $h_{\max}^{it} < h \leq h_{\max}$, $E_h^{g^{it}} = E_h^g$ and $d_{g^{it}}(h) = \tau^*(\alpha)$.

- $h_{\max} < h_{\max}^{it}$**:** If $x \in E_h^{g^{it}}$ with $\gamma < h \leq h_{\max}$, the same argument as above show that $d_{g^{it}}(h) = \max(\tau^*(\alpha), \tau^*(\omega_{it}^{-1}(h)))$.
If $h_{\max} < h \leq h_{\max}^{it}$, $E_h^{g^{it}} = \cup_{\alpha' \leq \omega_{it}^{-1}(h)} E_{\alpha', \omega_{it}^{-1}(h)}$ and $d_{g^{it}}(h) = \tau^*(\omega_{it}^{-1}(h))$.
In both cases, formula (5.3) gives the multifractal spectrum of g^{it}. □

Since ω_{it} is a linear decreasing function, the graph of the second function $h \to d_g(\omega_{it}^{-1}(h))$ on $[\gamma, h_{\max}^{it}]$ is a symmetric and dilated version of the graph of $h \to d_g(h)$ on $[h_{\min}, \gamma]$. In particular, when $h_{\max}^{it} > h_{\max}$, the multifractal formalism fails, since $d_{g^{it}}(h) > (\xi_g - 1)^*(h) = \tau^*(\alpha)$ (with $h = s_0 - 1/p_0 + \alpha/p_0$) (see Figure 3 for a plot of the multifractal spectrum of g^{it}).

5.2. **Multifractal spectrum of g^t.** Let $\gamma \in [h_{\min}, h_{\max}]$. g^t is obtained after applying a threshold of order γ to g as in Definition 3.3: $g^t = \sum_{j,k} d_{j,k} \mathbf{1}_{|d_{j,k}| \geq 2^{-j\gamma}} \psi_{j,k}$.

Theorem 5.2. *Let $\omega_t : [\gamma, h_{\max}] \to (0, +\infty)$ be the increasing function*

$$u \to \gamma \frac{u - (s_0 - 1/p_0) + \log_2 q_1/p_0}{\gamma - (s_0 - 1/p_0) + \log_2 q_1/p_0}.$$

Let $h_{\max}^t = \omega_t^{-1}(h_{\max})$. The multifractal spectrum of g^t ranges in $[h_{\min}, h_{\max}^t]$, and equals

$$d_{g^t}(h) = \begin{cases} d_g(h) & \text{if} \quad h \in [h_{\min}, \gamma], \\ d_g(\omega_t^{-1}(h)) & \text{if} \quad h \in (\gamma, h_{\max}^t]. \end{cases}$$

The proof is similar to the one of Theorem 5.1.

5.3. **Failure of the multifractal formalisms for g^t and g^{it}.** Remark that the Hausdorff multifractal spectrum of g^t (resp. g^{it}) is homogeneous in the sense that the series g^t (resp. g^{it}) has the same spectrum on any non-trivial subinterval of $[0, 1]$.

• Let us first discuss the case of g^t.

The Hausdorff multifractal spectra of g^t may be non convex, according to the choice $(q_0, q_1, s_0, p_0, \gamma)$. This happens as soon as $h_{\min} < \gamma < h_{\max}$, and in this case the multifractal formalism for g^t, as defined in Definition 2.3, is not satisfied when $h \in (\gamma, h_{\max}^t]$ (one has $\tilde{d}_{g^t}(h) = 0$ for $h > \gamma$). This is easily shown by computing the scaling function of associated with g^t, which is achieved as follows.

By definition of the Legendre transform of ξ^g, there exists $p_\gamma \in \mathbb{R}$ such that $(\xi_g)'(p_\gamma) = \gamma$ and $(\xi^g)^*(\gamma) = p_\gamma \gamma - \xi^g(p_\gamma)$ (indeed, $(\xi^g)^*(\gamma) = \inf_{p\in\mathbb{R}}(p\gamma - \xi^g(p))$ and the infimum is reached for $p = p_\gamma$). Since the coefficients $d_{j,k}$ which verify $|d_{j,k}| \geq 2^{-j\gamma}$ are left unchanged after threshold, one has that for every $p \geq p_\gamma$, $\xi^g(p) = \xi^{g^t}(p)$. This is equivalent, by Legendre transform, to the fact that the left part of the spectrum (i.e. the range of h's such that $h_{\min} \leq h \leq \gamma$) is left unchanged by the threshold of order γ.

Let now us compute ξ^{g^t} for $p < p_\gamma$ (this amounts to focusing on the right part of the spectrum d_{g^t}, i.e. $d_{g^t}(h)$ when $h > \gamma$). Since there is no more coefficients at scale j smaller than $2^{-j\gamma}$, the greatest contributions in the sum $\sum_{k=0}^{2^j-1} |d_{j,k}|^p$ come from the coefficients of order $\sim 2^{-j\gamma}$. An easy computation shows that

$$\text{for any } p < p_\gamma\,, \quad \sum_{k=0}^{2^j-1} |d_{j,k}|^p \sim 2^{-j(\xi^g(p_\gamma)+(p-p_\gamma)\gamma)}.$$

Hence, for $p < p_\gamma$, $\xi^{g^t}(p) = \xi^g(p_\gamma) + (p - p_\gamma)\gamma$. As a conclusion, by Legendre transform of ξ^{g^t}, the Legendre spectrum of g^t is

$$\begin{aligned} \text{if } h_{\min} \leq h \leq \gamma\,, \qquad & (\xi^{g^t})^*(h) = (\xi^g)^*(h), \\ \text{if } h > \gamma\,, \qquad & (\xi^{g^t})^*(h) = -\infty. \end{aligned}$$

This implies that the support of $(\xi^{g^t})^*$ is exactly $[h_{\min}, \gamma]$. Hence the multifractal formalism fails for g^t for the exponents $h \in [\gamma, h^t_{\max}]$.

Theorem 3.2 confirms a posteriori the presence of oscillating singularities in every $E_h^{g^t}$, for $h \in (\gamma, h^t_{\max}]$.

• The case of the series g^{it} is very comparable to the series g^t above. One obtains that, if p_γ is again the unique $p \in \mathbb{R}$ such that $(\xi_g)'(p) = \gamma$, then

$$\begin{aligned} \text{if } h < \gamma\,, \qquad & (\xi^{g^{it}})^*(h) = -\infty, \\ \text{if } \gamma \leq h \leq h_{\max}\,, \qquad & (\xi^{g^{it}})^*(h) = (\xi^g)^*(h). \end{aligned}$$

Hence the multifractal formalism fails as soon as $h^{it}_{\max} > h_{\max}$, since in this case, for every $h \in [h_{\max}, h^{it}_{\max}]$, $(\xi^{g^{it}})^*(h) > d_{g^{it}}(h)$. Remark that the failure (or not) of the formalism for g^{it} clearly depends on the choice of $\gamma \in (h_{\min}, h_{\max})$, while this is not the case for the regular threshold (the formalism is never satisfied for g^t as soon as $\gamma \in (h_{\min}, h_{\max})$).

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