

On sojourn of Brownian motion inside moving boundaries

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Abstract

We investigate the large scale structure of certain sojourn sets of one dimensional Brownian motion within two-sided moving boundaries. The macroscopic Hausdorff dimension, upper mass dimension and logarithmic density of these sets, are computed. We also give a uniform macroscopic dimension result for the Brownian level sets.

Keywords: Macroscopic Hausdorff dimension, mass dimension, Brownian motion, sojourn sets

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1. Introduction

Let $B = \{B_t, t \geq 0\}$ be a real-valued standard Brownian motion starting from 0. This article is concerned with the sojourn properties of B within some moving boundaries. More precisely, for an appropriate function $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, consider the sets

$$E(\varphi) = \{t \geq 0 : |B_t| \leq \varphi(t)\}. \quad (1)$$

that we call the set of Brownian sojourn within the two-sided boundary $\varphi(\cdot)$.

Besides its obvious application in physics and finance, the understanding of these sets at different scales entails considerable information on the path properties of the Brownian motion. Two types of study are of particular interest.

- Geometric properties of $E(\varphi)$ near $t_0 = 0$. This corresponds to the regular behavior of a Brownian path near zero. Concretely, local asymptotic laws of the Brownian motion such as Khintchine's law of iterated logarithm can be

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described in terms of geometric properties of these sets around 0 with specific choices for φ . A natural question is under which condition on φ these sets admit an upper density with respect to the Lebesgue measure (denoted by $|\cdot|$ throughout the paper), namely,

$$\limsup_{s \rightarrow 0} \frac{|E(\varphi) \cap [0, s]|}{s} = c_\varphi > 0.$$

Uchiyama [19] treated the case $\varphi(t) = h(t)\sqrt{t}$ where h is taken from a whole class of correction functions (of logarithmic order), and he provided an integral test for c_φ to be positive.

- Geometric properties of $E(\varphi)$ at infinity. This is related to the long time behavior of the Brownian motion. As B behaves like a square root function at infinity, the set $E(\varphi)$, when φ grows slower than the square root function, concerns the lower than normal growth of the Brownian motion. Uchiyama [19] established upper density bounds for $\varphi(t) = \sqrt{t}/h(t)$ with h belonging to a large class of correction functions with logarithmic order.

Let us mention briefly other related works. Consider

$$\tilde{E}(\varphi) = \{t \geq 0 : |B_t| \geq \varphi(t)\}.$$

where φ grows like a square root function with a logarithmic order correction. The geometry of these sets around zero describes the local irregular behavior of B , whereas their geometry around infinity describes the distribution of high peaks of B - we refer the interested reader to Strassen [17] and Uchiyama [19] for upper density results (around zero and infinity). We also mention the recent work of Khoshnevisan, Kim and Xiao [10] and Khoshnevisan and Xiao [11] who considered, among other things, the high peaks of symmetric stable Lévy processes. We refer to [7, 15, 4, 5, 3] for other results on sojourn properties of various stochastic processes.

Motivated by these studies, we focus on the asymptotic behavior around infinity of the sojourn sets of Brownian motion within moving boundaries with *much lower than normal growth*. For this, we introduce the sets

$$\forall \gamma \in [0, 1/2], \quad E_\gamma := E(\varphi_\gamma) \text{ with } \varphi_\gamma(t) = t^\gamma, \quad t \geq 0,$$

and our goal is to estimate the size of E_γ for all values of γ . This size is expressed in terms of three different quantities, including the notion of large scale dimensions developed by Barlow and Taylor [1, 2] in the late 80's, recently "refreshed" in the work of Xiao and Zheng [20], Georgiou *et al.* [8], Khoshnevisan *et al.* [10, 11].

The initial motivation of Barlow and Taylor was to define a notion of fractals in discrete spaces such as \mathbb{Z}^d in order to characterize the size of models in statistical physics, e.g. the infinite connected component of a percolation process, the range of a transient random walk etc. To this end, they introduced and investigated several notions of dimension describing different types of asymptotic

behavior of a set around infinity. Each dimension corresponds to an analog in large scales of a classical fractal dimension. Among these dimensions, we are going to use the “macroscopic Hausdorff dimension” and the “mass dimensions”, which are analogs of the classical Hausdorff dimension and Minkowski dimensions in large scales, respectively.

Let us state our first main theorem. The macroscopic Hausdorff dimension of a set $E \subset \mathbb{R}^d$ is denoted by $\text{Dim}_H E$, the upper mass dimension by $\text{Dim}_{UM} E$, and the logarithmic density by Den_{\log} . The definitions are recalled in Section 2.

Theorem 1. *Almost surely, for all $\gamma \in [0, \frac{1}{2}]$,*

$$\text{Dim}_H E_\gamma = \begin{cases} \frac{1}{2} & \text{if } \gamma \in [0, \frac{1}{2}), \\ 1 & \text{if } \gamma = \frac{1}{2}. \end{cases} \quad (2)$$

$$\text{Dim}_{UM} E_\gamma = \text{Den}_{\log} E_\gamma = \frac{1}{2} + \gamma. \quad (3)$$

Remark 1.

- *Since the standard scale of Brownian motion at time t is \sqrt{t} and Brownian paths fluctuate not so violently in the scale \sqrt{t} , it is expected that the fractal dimension of E_γ does not depend on γ if $\gamma < \frac{1}{2}$. This intuition is confirmed by (2) and can be interpreted as such, B spends most of the time at its “boundary” from the macroscopic Hausdorff dimension standpoint.*
- *Although they have the same macroscopic Hausdorff dimension, the sets E_γ , $0 \leq \gamma < 1/2$, differ by their upper mass dimensions. Indeed, (3) indicates a multifractal behavior for $\text{Dim}_{UM} E_\gamma$, and gives a natural example of sets for which the macroscopic Hausdorff dimension and the upper mass dimension differ.*
- *The Brownian motion B returns inside the boundary φ_γ infinitely many times, for every $\gamma \in [0, 1/2]$, so the sets E_γ are unbounded. Our result quantifies recurrence and fluctuation properties of B at large scales.*

Our contribution to the formula (2) is the equality when $\gamma \in [0, 1/2)$. The case $\gamma = \frac{1}{2}$ is deduced from Theorem 2 in [19] where Uchiyama obtained that

$$\text{a.s.} \quad \limsup_{r \rightarrow +\infty} \frac{|E(\frac{1}{2}) \cap [0, r]|}{r} > 0.$$

This inequality entails that a.s. $\text{Dim}_H E(\frac{1}{2}) = 1$ thanks to the following fact proved in [10] : for any $E \subset \mathbb{R}$,

$$\limsup_{r \rightarrow +\infty} \frac{|E \cap [0, r]|}{r} > 0 \quad \Rightarrow \quad \text{Dim}_H E = 1.$$

Our second result is a uniform dimension result for the level sets

$$\mathcal{Z}_x = \{t \geq 0 : B_t = x\}.$$

It turns out that, though much thinner than any E_γ , the level sets all have the same macroscopic Hausdorff dimension as that of E_γ , for $0 < \gamma < \frac{1}{2}$.

Theorem 2. *Almost surely, for all $x \in \mathbb{R}$, $\text{Dim}_H(\mathcal{Z}_x) = \frac{1}{2}$.*

The local structure of \mathcal{Z}_x is well understood since the works by Taylor and Wendel [18] and Perkins [14], who proved Hausdorff measure results for \mathcal{Z}_x using local times for fixed x and for all x , respectively. In particular, the classical Hausdorff dimension of \mathcal{Z}_0 is $\frac{1}{2}$, a result already known in [6]. Our result gives the large scale structure of \mathcal{Z}_x and might be compared with an interesting result by Khoshnevisan [9] which states that the zero set of a symmetric random walk $\{\xi_n : n \in \mathbb{N}\}$ in \mathbb{Z}^1 with finite variance

$$\{n \in \mathbb{N} : \xi_n = 0\}$$

has macroscopic Hausdorff dimension $\frac{1}{2}$.

This paper is organized as follows. In Section 2, we recall the definition of large scale dimensions and establish some preliminary estimates. Theorem 1 is proved in Section 3. Theorem 2 is proved in Section 4.

2. Preliminaries

Throughout the paper, c, C are generic positive finite constants whose value may change from line to line. For two families of positive real numbers $(a(x))$ and $(b(x))$, the equation $a(x) \asymp b(x)$ means that the ratio $a(x)/b(x)$ is uniformly bounded from below and above by some positive finite constant independent of x . The set of non-negative real numbers is denoted by \mathbb{R}^+ and the set of positive integers by \mathbb{N}^* . Also, \mathbb{P}^x denotes the law of Brownian motion starting from $x \in \mathbb{R}$, and \mathbb{E}^x denotes the expectation with respect to \mathbb{P}^x . For simplicity, we write $\mathbb{P} = \mathbb{P}^0$ and $\mathbb{E} = \mathbb{E}^0$.

2.1. Macroscopic dimensions

We adopt the notations in [10, 11] that we recall now. Set $Q(x, r) = [x, x+r)$ for all $x \in \mathbb{R}$ and $r > 0$.

Define the annuli $\forall n \geq 1, \mathcal{S}_n = [2^{n-1}, 2^n)$ and $\mathcal{S}_0 = [0, 1)$. For any $\rho \geq 0$, any set $E \subset \mathbb{R}^+$, $n \in \mathbb{N}^*$, we introduce the quantity

$$\nu_\rho^n(E) = \inf \left\{ \sum_{i=1}^m \left(\frac{|Q_i|}{2^n} \right)^\rho : Q_i \subset \mathcal{S}_n, E \cap \mathcal{S}_n \subset \bigcup_{i=1}^m Q_i, \right. \\ \left. \text{and } Q_i \text{ is a non-trivial interval with integer boundaries} \right\}. \quad (4)$$

Other gauge functions could be used instead of $x \mapsto x^\rho$.

The key constraint is that the sets Q_i used in (4) are non-trivial intervals with integer bounds. In particular, their length is always greater than 1.

Definition 1. Let $E \subset \mathbb{R}^+$. The macroscopic Hausdorff dimension of E is defined as

$$\text{Dim}_{\text{H}}E = \inf \left\{ \rho \geq 0 : \sum_{n \geq 0} \nu_{\rho}^n(E) < +\infty \right\}. \quad (5)$$

We move to the definition of mass dimensions and logarithmic density. For any $E \subset \mathbb{R}^+$, define the pixelization of E by the set of integers that are of distance at most 1 from E , namely,

$$\text{pix}(E) = \{x \in \mathbb{Z} : \text{dist}(x, E) \leq 1\}.$$

Definition 2. The upper and lower mass dimension of E are defined as

$$\begin{aligned} \text{Dim}_{\text{UM}}E &= \limsup_{n \rightarrow +\infty} \frac{\log_2(\#\text{pix}(E \cap [0, 2^n]))}{n}, \\ \text{Dim}_{\text{LM}}E &= \liminf_{n \rightarrow +\infty} \frac{\log_2(\#\text{pix}(E \cap [0, 2^n]))}{n}. \end{aligned}$$

The logarithmic density of E is defined as

$$\text{Den}_{\log}E = \limsup_{n \rightarrow +\infty} \frac{\log_2 |E \cap [0, 2^n]|}{n}.$$

Recall that $|A|$ stands for the Lebesgue measure of A . The macroscopic Hausdorff dimension of a set does not depend on any of its bounded subsets, since the series in (5) converges if and only if its tail series converges. Further, the covering intervals are chosen to have length larger than 1, which explains why the macroscopic Hausdorff dimension does not rely on the local structure of the underlying set. The following fact is known [1, 2]: for any set $E \subset \mathbb{R}$,

$$\text{Dim}_{\text{H}}E \leq \text{Dim}_{\text{LM}}E \leq \text{Dim}_{\text{UM}}E.$$

Since the cardinal of the pixelization of a set E is always larger than the Lebesgue measure of E , we also have

$$\text{Den}_{\log}E \leq \text{Dim}_{\text{UM}}E.$$

To bound $\text{Dim}_{\text{H}}E$ from above, one usually exhibits an economic covering of E . To get a lower bound, the following lemma, which is an analog of the mass distribution principle, is useful.

Lemma 1. Let $E \subset \mathcal{S}_n$, $n \in \mathbb{N}$. Let μ be a finite Borel measure on \mathbb{R} with support included in E . Suppose that there exists finite positive constants C and ρ , such that for any interval $Q \subset \mathcal{S}_n$ with $|Q| \geq 1$, one has

$$\mu(Q) \leq C|Q|^{\rho}.$$

Then

$$\nu_{\rho}^n(E) \geq C^{-1}2^{-n\rho} \mu(\mathcal{S}_n).$$

In fact, there is more flexibility in the choice of covering intervals when we are only concerned with the value of $\text{Dim}_H E$. Let us introduce, for every integer $n \geq 1$ and any set $E \subset \mathbb{R}^+$, the quantity

$$\tilde{\nu}_\rho^n(E) = \inf \left\{ \sum_{i=1}^m \left(\frac{|Q_i|}{2^n} \right)^\rho : Q_i \subset \mathcal{S}_n, E \cap \mathcal{S}_n \subset \bigcup_{i=1}^m Q_i, \frac{|Q_i|}{n^2} \in \mathbb{N}^*, \right. \\ \left. \text{and } Q_i \text{ is an interval with integer boundaries} \right\}. \quad (6)$$

The difference with ν_ρ^n is that coverings by sets of size that are multiple of n^2 are used, and this does not change the value of the macroscopic Hausdorff dimension, as stated by the following lemma.

Lemma 2. *For every set $E \subset \mathbb{R}^+$,*

$$\text{Dim}_H E = \inf \left\{ \rho \geq 0 : \sum_{n \geq 0} \tilde{\nu}_\rho^n(E) < +\infty \right\}. \quad (7)$$

Proof. Let us denote by \tilde{d} the value in the right hand-side of (7).

Obviously, for every $\rho \geq 0$, $\tilde{\nu}_\rho^n(E) \geq \nu_\rho^n(E)$. Hence, recalling (5), $\text{Dim}_H E \leq \tilde{d}$. If $\text{Dim}_H(E) = 1$, then (7) holds trivially. Assume thus $\text{Dim}_H(E) < 1$.

Let $1 > \rho' > \rho > \text{Dim}_H(E)$, and fix $n \geq 1$. Choose $\mathcal{Q}_n := \{Q_i\}_{i=1, \dots, m}$ as a finite family of intervals such that $E \cap \mathcal{S}_n \subset \bigcup_{i=1}^m Q_i$, with $|Q_i| \geq 1$ and $\sum_{i=1}^m \left(\frac{|Q_i|}{2^n} \right)^\rho \leq 2\nu_\rho^n(E)$.

Define the finite family of intervals $\tilde{\mathcal{Q}}_n := \{\tilde{Q}_i\}_{i=1, \dots, m}$ as follows: \tilde{Q}_i is an interval containing Q_i , included in \mathcal{S}_n , whose length is the smallest possible multiple of n^2 .

Observe that if $|Q_i| \geq n^2$, $|\tilde{Q}_i| \leq 2|Q_i|$, while if $1 \leq |Q_i| < n^2$, $|\tilde{Q}_i| \leq n^2|Q_i|$.

By construction, $E \cap \mathcal{S}_n \subset \bigcup_{i=1}^m \tilde{Q}_i$. In addition, since $\rho < \rho' < 1$,

$$\begin{aligned} \sum_{i=1}^m \left(\frac{|\tilde{Q}_i|}{2^n} \right)^{\rho'} &= \sum_{i=1: |Q_i| < n^2}^m \left(\frac{|\tilde{Q}_i|}{2^n} \right)^{\rho'} + \sum_{i=1: |Q_i| \geq n^2}^m \left(\frac{|\tilde{Q}_i|}{2^n} \right)^{\rho'} \\ &\leq \sum_{i=1: |Q_i| < n^2}^m n^2 \left(\frac{|Q_i|}{2^n} \right)^{\rho'} + 2 \sum_{i=1: |Q_i| \geq n^2}^m \left(\frac{|Q_i|}{2^n} \right)^{\rho'} \\ &\leq \frac{n^{2+\rho'-\rho}}{2^{n(\rho'-\rho)}} \sum_{i=1: |Q_i| < n^2}^m \left(\frac{|Q_i|}{2^n} \right)^\rho + 2 \sum_{i=1: |Q_i| \geq n^2}^m \left(\frac{|Q_i|}{2^n} \right)^\rho. \end{aligned}$$

When n becomes large, $\frac{n^{2+\rho'-\rho}}{2^{n(\rho'-\rho)}} \leq 1$, hence

$$\sum_{i=1}^m \left(\frac{|\tilde{Q}_i|}{2^n} \right)^{\rho'} \leq 2 \sum_{i=1}^m \left(\frac{|Q_i|}{2^n} \right)^\rho.$$

One deduces that $\tilde{\nu}_{\rho'}^n(E) \leq 2\nu_{\rho}^n(E)$. Since $\sum_{n \geq 1} \nu_{\rho}^n(E) < +\infty$, the series $\sum_{n \geq 1} \tilde{\nu}_{\rho'}^n(E)$ also converges, and $\tilde{d} \leq \rho'$. Letting ρ' tend to $\text{Dim}_{\text{H}}(E)$ yields the result. \square

Remark 2. *The same argument shows that for any $C \geq 1$, using coverings of a set $E \cap \mathcal{S}_n$ by sets of size larger than C or than Cn^2 instead of 1 or n^2 in the definitions (4) or (6), does not change the value of the macroscopic Hausdorff dimension of E .*

2.2. Hitting probability estimates of Brownian motion inside the moving boundaries

The following estimate is useful when looking for an appropriate covering of E_γ with respect to different large scale dimensions.

Lemma 3. *Consider an interval $Q(a, r)$ inside \mathcal{S}_n , i.e. $a \geq 2^{n-1}$ and $a+r \leq 2^n$. For each $0 \leq \gamma < 1/2$, define the event*

$$\mathcal{A}(n, a, r, \gamma) = \{\exists t \in Q(a, r) : |B_t| \leq t^\gamma\}. \quad (8)$$

One has

$$\mathbb{P}(\mathcal{A}(n, a, r, \gamma)) \leq \frac{2}{\sqrt{\pi}} 2^{n(\gamma-1/2)} + \frac{4}{\sqrt{2\pi}} \left(\frac{r}{a}\right)^{1/2}.$$

Following a standard vocabulary, the event $\mathcal{A}(n, a, r, \gamma)$ describes the hitting probability of B inside the moving boundary φ_γ . Lemma 3 states that, when $\frac{r}{a}$ is small, this hitting probability depends only on γ and n , while it behaves like $\left(\frac{r}{a}\right)^{1/2}$ when $\frac{r}{a}$ becomes large. Basic properties of Brownian motion used in the proof of Lemma 3 can be found in [16] or [13].

Proof of Lemma 3: One has

$$\begin{aligned} \mathbb{P}(\mathcal{A}(n, a, r, \gamma)) &\leq \mathbb{P}\left(\inf_{t \in Q(a, r)} |B_t| \leq 2^{n\gamma}\right) \\ &= \mathbb{P}(|B_a| \leq 2^{n\gamma}) + \mathbb{P}\left(|B_a| > 2^{n\gamma}, \inf_{t \in Q(a, r)} |B_t| \leq 2^{n\gamma}\right) \\ &:= P_1 + P_2. \end{aligned}$$

By the self-similarity of B and recalling that $a \in \mathcal{S}_n$, one obtains

$$P_1 = \mathbb{P}(|B_1| \leq a^{-1/2} 2^{n\gamma}) \leq \sqrt{\frac{2}{\pi}} a^{-1/2} 2^{n\gamma} \leq \frac{2}{\sqrt{\pi}} 2^{n(\gamma-1/2)}.$$

Using the symmetry of B , one gets

$$\mathbb{P}\left(B_a > 2^{n\gamma}, \inf_{t \in Q(a, r)} B_t \leq 2^{n\gamma}\right) = \mathbb{P}\left(B_a < -2^{n\gamma}, \sup_{t \in Q(a, r)} B_t \geq -2^{n\gamma}\right).$$

Thus,

$$\begin{aligned} P_2 &\leq 2\mathbb{P}\left(B_a > 2^{n\gamma}, \inf_{t \in Q(a,r)} B_t \leq 2^{n\gamma}\right) \\ &= 2\mathbb{P}\left(B_a > 2^{n\gamma}, \inf_{t \in Q(a,r)} (B_t - B_a) \leq 2^{2n\gamma} - B_a\right). \end{aligned}$$

Set $\tilde{B}_h = B_{a+h} - B_a$ which is a Brownian motion independent of B_a . Using successively the self-similarity, the symmetry and the Markov property yields

$$\begin{aligned} P_2 &\leq 2\mathbb{P}\left(B_a > 2^{n\gamma}, \inf_{0 \leq h \leq r} \tilde{B}_h \leq 2^{n\gamma} - B_a\right) \\ &= 2\mathbb{P}\left(B_a > 2^{n\gamma}, \inf_{0 \leq h \leq 1} \tilde{B}_h \leq r^{-1/2}(2^{n\gamma} - B_a)\right) \\ &= 2\mathbb{P}\left(B_a > 2^{n\gamma}, \sup_{0 \leq h \leq 1} \tilde{B}_h \geq r^{-1/2}(B_a - 2^{n\gamma})\right) \\ &= 2 \int_{2^{n\gamma}}^{+\infty} \mathbb{P}\left(\sup_{0 \leq h \leq 1} \tilde{B}_h \geq r^{-1/2}(x - 2^{n\gamma})\right) e^{-\frac{x^2}{2a}} \frac{dx}{\sqrt{2\pi a}} := I_1 + I_2, \end{aligned}$$

where I_1 is the integral over the interval $[2^{n\gamma}, 2^{n\gamma} + r^{1/2}]$, and I_2 over $[2^{n\gamma} + r^{1/2}, \infty)$. On one hand, bounding from above the probability in the integrand by 1, one obtains

$$I_1 \leq 2 \int_{2^{n\gamma}}^{2^{n\gamma} + r^{1/2}} e^{-\frac{x^2}{2a}} \frac{dx}{\sqrt{2\pi a}} \leq \frac{2}{\sqrt{2\pi}} \left(\frac{r}{a}\right)^{1/2}.$$

On the other hand, one knows by applying the reflection principle to \tilde{B} that $\sup_{0 \leq h \leq 1} \tilde{B}_h$ has the same distribution as $|\tilde{B}_1|$. Hence, using the tail estimate for a standard normal random variable [12, page 192], one has

$$\begin{aligned} I_2 &\leq 4 \int_{2^{n\gamma} + r^{1/2}}^{+\infty} \mathbb{P}\left(\tilde{B}_1 \geq r^{-1/2}(x - 2^{n\gamma})\right) e^{-\frac{x^2}{2a}} \frac{dx}{\sqrt{2\pi a}} \\ &\leq 4 \int_{2^{n\gamma} + r^{1/2}}^{+\infty} \frac{1}{r^{-1/2}(x - 2^{n\gamma})\sqrt{2\pi}} e^{-\frac{(x-2^{n\gamma})^2}{2r}} e^{-\frac{x^2}{2a}} \frac{dx}{\sqrt{2\pi a}} \\ &\leq 4 \frac{1}{\sqrt{2\pi}} \int_{2^{n\gamma}}^{+\infty} e^{-\frac{(x-2^{n\gamma})^2}{2r}} \frac{dx}{\sqrt{2\pi a}} = \frac{4}{\sqrt{2\pi}} \left(\frac{r}{a}\right)^{1/2} \int_{2^{n\gamma}}^{+\infty} e^{-\frac{(x-2^{n\gamma})^2}{2r}} \frac{dx}{\sqrt{2\pi r}} \\ &= \frac{2}{\sqrt{2\pi}} \left(\frac{r}{a}\right)^{1/2}. \end{aligned}$$

Therefore, one has established that

$$P_2 \leq \frac{4}{\sqrt{2\pi}} \left(\frac{r}{a}\right)^{1/2}.$$

Combining the estimates above ends the proof. \square

2.3. Brownian local times

Let us recall very briefly the notion of local times. Let f be a non-negative continuous even function with integral one. For each $\varepsilon > 0$, set $f_\varepsilon(x) = f(x/\varepsilon)/\varepsilon$. Define the local times at zero of B by

$$L_t = \lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \int_0^t f_\varepsilon(B_s) ds.$$

It is known [12, Lemma 2.4.1] that the convergence occurs uniformly on $t \in [0, T]$, \mathbb{P}^y almost surely, for any $y \in \mathbb{R}$ and $T > 0$. From the definition we see that $t \mapsto L_t$ extends to a measure, denoted by $L(dt)$, which is supported on the zero set of Brownian motion [12, Remark 3.6.2].

Next lemma gives the asymptotic behavior at infinity of the local time increments, which is the continuous analog of Corollary 2.2 in [9].

Lemma 4. *One has \mathbb{P} -a.s.*

$$\limsup_{n \rightarrow \infty} n^{-1} \sup_{\substack{t \leq 2^n \\ t \in \mathbb{N}}} \sup_{\substack{2 \leq h \leq 2^{n-2} \\ h \in \mathbb{N}}} \frac{L_{t+h} - L_t}{\sqrt{h/\log_2 h}} \leq 4.$$

Proof. Set $S_t = \sup_{0 \leq s \leq t} B_s$. A famous theorem by Lévy [16, page 240] states that the processes $\{\bar{S}_t, t \geq 0\}$ and $\{L_t, t \geq 0\}$ have the same law. Also, by the reflection principle, $S_t = |B_t|$ in distribution for any fixed t . Therefore, the classical Gaussian tail estimate [12, page 192] yields that for all $t, x > 0$,

$$\mathbb{P}(L_t \geq x) = \mathbb{P}(|B_1| \geq x/\sqrt{t}) \leq e^{-x^2/(2t)}. \quad (9)$$

For each $t \geq 0$, define $T_t = \inf\{s \geq t : B_s = 0\}$ the first hitting time at zero of Brownian motion after time t . Since L only increases when the Brownian motion hits zero, one has for almost every sample path ω that, $L_{t+h} - L_t \leq L_{T_t+h} - L_{T_t} = L_h \circ \theta_{T_t}$ for any $t, h \geq 0$, see [16, page 402] for the equality. Here θ is the usual shift operator on the space of continuous functions.

This, combined with (9) and the strong Markov property at the stopping time T_t , implies that for every $x > 0$,

$$\begin{aligned} \mathbb{P} \left(\sup_{\substack{t \leq 2^n \\ t \in \mathbb{N}}} \sup_{\substack{2 \leq h \leq 2^{n-2} \\ h \in \mathbb{N}}} \frac{L_{t+h} - L_t}{\sqrt{h/\log_2 h}} \geq x \right) &\leq 2^{2n-2} \sup_{\substack{t \leq 2^n \\ 2 \leq h \leq 2^{n-2}}} \mathbb{P} \left(\frac{L_{t+h} - L_t}{\sqrt{h/\log_2 h}} \geq x \right) \\ &\leq 2^{2n-2} \sup_{2 \leq h \leq 2^{n-2}} \mathbb{P} \left(L_h \geq x \sqrt{h/\log_2 h} \right) \\ &\leq 2^{2n} e^{-x^2/(2n)} \end{aligned}$$

Taking $x = 4n$ in the above inequality, and summing over $n \geq 1$, the Borel-Cantelli lemma yields the conclusion. \square

3. Proof of Theorem 1

3.1. Macroscopic Hausdorff dimension of E_γ

We prove the dimension formula (2). As said in the introduction, by the upper density result of Uchiyama [19], it is enough to compute $\text{Dim}_\mathbb{H} E_\gamma$ for all $\gamma \in [0, \frac{1}{2})$. Let $0 \leq \gamma < \frac{1}{2}$ be fixed throughout this section. Due to the monotonicity in γ of the sets E_γ , and the fact that the zero set of Brownian motion is included in E_0 , we divide the proof of (2) into two parts :

$$\text{Dim}_\mathbb{H} E_\gamma \leq \frac{1}{2} \quad \text{and} \quad \text{Dim}_\mathbb{H} \mathcal{Z}_0 \geq \frac{1}{2}.$$

3.1.1. Proof of $\text{Dim}_\mathbb{H} E_\gamma \leq \frac{1}{2}$

A first moment argument is used for this upper bound. Let $\rho > \frac{1}{2}$ and set

$$x_{n,i} = 2^{n-1} + i2^{2n\gamma} \text{ for } i \in \{0, \dots, \lfloor 2^{n-1}/2^{2n\gamma} \rfloor\}.$$

Consider the intervals $Q(x_{n,i}, 2^{2n\gamma})$ which form a partition of \mathcal{S}_n . Note that $E_\gamma \cap Q(x_{n,i}, 2^{2n\gamma}) \neq \emptyset$ if and only if the event $\mathcal{A}(n, x_{n,i}, 2^{2n\gamma}, \gamma)$ occurs (see its definition in Lemma 3). Thus,

$$\nu_\rho^n(E_\gamma) \leq \sum_{i=0}^{\lfloor 2^{(1-2\gamma)n-1} \rfloor} \left(\frac{2^{2n\gamma}}{2^n} \right)^\rho \mathbf{1}_{\mathcal{A}(n, x_{n,i}, 2^{2n\gamma}, \gamma)}.$$

By choosing the length $2^{2n\gamma}$, one observes that the two terms in Lemma 3 are of the same order. Taking expectation in the above inequality, one obtains by Lemma 3 that there exists a positive finite constant C such that for all $n \in \mathbb{N}^*$

$$\begin{aligned} \mathbb{E}[\nu_\rho^n(E_\gamma)] &\leq 2^{(1-2\gamma)(1-\rho)n} \cdot C2^{n(\gamma-1/2)} \\ &= C2^{(1/2-\gamma)(1-2\rho)n}. \end{aligned}$$

Thus, the Fubini Theorem entails $\mathbb{E}[\sum_{n=1}^\infty \nu_\rho^n(E_\gamma)] < +\infty$. This proves that $\text{Dim}_\mathbb{H} E_\gamma \leq \rho$ almost surely. Letting $\rho \downarrow 1/2$ yields the upper bound.

3.1.2. Proof of $\text{Dim}_\mathbb{H} \mathcal{Z}_0 \geq 1/2$

The proof follows the idea of Khoshnevisan [9, page 581].

Let $g(\varepsilon) = \sqrt{\varepsilon \log_2(1/\varepsilon)}$ for $0 < \varepsilon \leq 1/2$. By Lemma 4, there exists a random integer $n_0(\omega)$ such that for all $n \geq n_0(\omega)$,

$$\sup_{\substack{t \leq 2^n \\ t \in \mathbb{N}}} \sup_{\substack{2 \leq h \leq 2^{n-2} \\ h \in \mathbb{N}}} \frac{L_{t+h} - L_t}{\sqrt{h/\log_2 h}} \leq 5n. \quad (10)$$

Consider any finite sequence of intervals $\{Q_{n,i} = [a_{n,i}, b_{n,i}]\}_{i=1, \dots, m}$ with integer endpoints and with sidelength $2 \leq |Q_{n,i}| \leq 2^{n-2}$, that form a covering of $\mathcal{Z}_0 \cap \mathcal{S}_n$.

Using (10), one obtains

$$\begin{aligned}
g\left(\frac{|Q_{n,i}|}{2^n}\right) &= \sqrt{\frac{|Q_{n,i}|}{2^n} \log_2 \frac{2^n}{|Q_{n,i}|}} \\
&= 2^{-n/2} \sqrt{\frac{|Q_{n,i}|}{\log_2 |Q_{n,i}|} (n - \log_2 |Q_{n,i}|) \log_2 |Q_{n,i}|} \\
&\geq (n-1)^{1/2} 2^{-n/2} \frac{L_{b_{n,i}} - L_{a_{n,i}}}{5n} \geq \frac{1}{6} n^{-1/2} 2^{-n/2} (L_{b_{n,i}} - L_{a_{n,i}}),
\end{aligned}$$

where we used the elementary inequality $\sqrt{x(n-x)} \geq \sqrt{n-1}$ for all $1 \leq x \leq n-1$. Therefore, a.s. for all $n \geq n_0(\omega)$ and any covering of $\mathcal{Z} \cap \mathcal{S}_n$ by a finite family of intervals $(Q_{n,i})_{i=1,\dots,m}$,

$$\sum_{i=1}^m g\left(\frac{|Q_{n,i}|}{2^n}\right) \geq \frac{L(\cup_i Q_{n,i})}{6\sqrt{n}2^n} = \frac{L_{2^n} - L_{2^{n-1}}}{6\sqrt{n}2^n},$$

where the last equality follows from the fact that the intervals $Q_{n,i}$ form a covering of the support of $L(dt)$ in \mathcal{S}_n . The desired lower bound for $\text{Dim}_{\mathbb{H}} \mathcal{Z}_0$ follows if we can show that a.s.

$$F_N = \sum_{n=1}^N \frac{L_{2^n} - L_{2^{n-1}}}{\sqrt{n}2^n} \tag{11}$$

diverges as $N \rightarrow +\infty$. Indeed, this would demonstrate that the sum (5) diverges for every $\rho < 1/2$ and for any choice of covering.

To this end, let us use a second moment argument. As for any fixed t , $L_t = |B_t|$ in distribution, one knows that

$$\mathbb{E}[L_t] = c\sqrt{t} \text{ and } \mathbb{E}[(L_t)^2] = t. \tag{12}$$

An Abel summation manipulation gives

$$F_N = \frac{L_{2^N}}{\sqrt{N}2^N} - \frac{L_2}{\sqrt{2}} + \sum_{n=2}^N L_{2^n} \left(\frac{1}{\sqrt{n}2^n} - \frac{1}{\sqrt{(n+1)2^{n+1}}} \right).$$

Thus,

$$\mathbb{E}[F_N] \asymp \sqrt{N}.$$

By Cauchy-Schwarz inequality, $\mathbb{E}[F_N^2] \geq cN$. To deduce an upper bound for $\mathbb{E}[F_N^2]$, we use the expression (11) and consider the expectation of the double sum F_N^2 . Recall that T_t is the first time the Brownian motion hits zero after time t and $L_{t+h} - L_t \leq L_{T_t+h} - L_{T_t} = L_h \circ \theta_{T_t}$. For $j < \ell$, applying the strong Markov property at $T_{2^{\ell-1}}$, then at $T_{2^{j-1}}$, implies

$$\mathbb{E}[(L_{2^j} - L_{2^{j-1}})(L_{2^\ell} - L_{2^{\ell-1}})] \leq \mathbb{E}[(L_{2^j} - L_{2^{j-1}})]\mathbb{E}[L_{2^{\ell-1}}] \leq \mathbb{E}[L_{2^{j-1}}]\mathbb{E}[L_{2^{\ell-1}}].$$

Also, we have

$$\mathbb{E}[(L_{2^n} - L_{2^{n-1}})^2] \leq \mathbb{E}[(L_{2^{n-1}})^2].$$

Hence, we obtain using (11) and (12)

$$\begin{aligned} \mathbb{E}[F_N^2] &= \sum_{n=1}^N \frac{\mathbb{E}[(L_{2^n} - L_{2^{n-1}})^2]}{n2^n} + 2 \sum_{\ell=1}^N \sum_{j=1}^{\ell-1} \frac{\mathbb{E}[(L_{2^j} - L_{2^{j-1}})(L_{2^\ell} - L_{2^{\ell-1}})]}{\sqrt{j2^j \ell 2^\ell}} \\ &\leq \sum_{n=1}^N \frac{1}{n} + 2c \sum_{\ell=1}^N \sum_{j=1}^{\ell-1} (j\ell)^{-1/2} \leq 3cN. \end{aligned}$$

Therefore, the Paley-Zygmund inequality yields that there are constants $c, c' > 0$ for which for every $N \geq 1$,

$$\mathbb{P}(F_N \geq c\sqrt{N}) \geq c'.$$

This proves $\mathbb{P}(F_\infty = \infty) > 0$. By Kolmogorov's zero-one law, $F_\infty = \infty$ almost surely, which completes the proof.

3.2. Upper mass dimension and logarithmic density of E_γ

In order to prove that $\text{Den}_{\log} E_\gamma = \text{Dim}_{\text{UM}} E_\gamma = \frac{1}{2} + \gamma$, by the relation $\text{Den}_{\log} E \leq \text{Dim}_{\text{UM}} E$ true for all set $E \subset \mathbb{R}^+$, it is enough to prove

$$\text{Dim}_{\text{UM}} E_\gamma \leq \frac{1}{2} + \gamma \quad \text{and} \quad \text{Den}_{\log} E_\gamma \geq \frac{1}{2} + \gamma.$$

3.2.1. Proof of $\text{Dim}_{\text{UM}} E_\gamma \leq \frac{1}{2} + \gamma$

We still combine the first moment argument with Lemma 3. For each \mathcal{S}_n , consider $Q(k, 1) \subset \mathcal{S}_n$ with $k \in \mathbb{N}$. Observe that $k \in \text{pix}(E_\gamma)$ implies that B is within the boundary E_γ on one of the time intervals $[k-1, k)$ and $[k, k+1)$. Therefore,

$$\#\text{pix}(E_\gamma \cap [0, 2^n]) \leq 2 \sum_{p=0}^n \sum_{k=2^{p-1}}^{2^p-1} \mathbf{1}_{\mathcal{A}(p, k, 1, \gamma)}.$$

Taking expectation, one obtains by Lemma 3 that for some finite positive constant C ,

$$\mathbb{E}[\#\text{pix}(E_\gamma \cap [0, 2^n])] \leq C \sum_{p=0}^n 2^p 2^{p(\gamma-1/2)} = C 2^{n(\gamma+1/2)}.$$

For $\rho > 1/2 + \gamma$, the Markov inequality yields that for all $n \geq 1$,

$$\mathbb{P}(\#\text{pix}(E_\gamma \cap [0, 2^n]) \geq 2^{n\rho}) \leq C^{-1} 2^{-n(\rho-\gamma-1/2)}$$

which is the general term of a convergent series. Thus, an application of Borel-Cantelli Lemma gives almost surely, for all n large enough,

$$\#\text{pix}(E_\gamma \cap [0, 2^n]) < 2^{n\rho}.$$

Hence, $\text{Dim}_{\text{UM}} E_\gamma \leq \rho$, almost surely. The desired upper bound follows by letting ρ tend to $\gamma + 1/2$.

3.2.2. *Proof of $\text{Den}_{\log} E_\gamma \geq \frac{1}{2} + \gamma$*

Define the sojourn times

$$M(t) = |\{0 < s < t : |B(s)| \leq s^\gamma\}|, \quad t > 0.$$

We need to show that

$$\limsup_{n \rightarrow \infty} \frac{\log_2 M(2^n)}{n} \geq \gamma + \frac{1}{2}, \quad \text{a.s.} \quad (13)$$

We apply a second moment argument. By Fubini's Theorem, we see that

$$\mathbb{E}[M(t)] = \int_0^t \mathbb{P}(|B(s)| \leq s^\gamma) ds = \int_0^t \mathbb{P}(|B_1| \leq s^{\gamma-\frac{1}{2}}) ds \asymp t^{\gamma+\frac{1}{2}}, \quad t \rightarrow +\infty. \quad (14)$$

To conclude, we shall use the following fact yet to be proved:

$$\limsup_{t \rightarrow \infty} \mathbb{E} \left[\left(M(t) t^{-(\gamma+\frac{1}{2})} \right)^2 \right] < \infty. \quad (15)$$

This, combined with (14) and the Paley-Zygmund inequality, leads to

$$\mathbb{P}(M(t) \geq c t^{\gamma+\frac{1}{2}}) > c'$$

for all n large enough with some positive constants c, c' . Owing to the triviality of the tail σ -field of Brownian motion, it follows that

$$\mathbb{P}(M(2^n) \geq c 2^{n(\gamma+\frac{1}{2})} \text{ infinitely often}) = 1,$$

which implies that

$$\limsup_{n \rightarrow \infty} \frac{M(2^n)}{2^{n(\gamma+\frac{1}{2})}} > 0, \quad \text{a.s.}$$

The desired lower bound (13) follows.

We end this section with the proof of (15). By Fubini's Theorem,

$$\begin{aligned} \mathbb{E}[M(t)^2] &= 2 \int_0^t \int_0^{t-a} \mathbb{P}(|B_a| \leq a^\gamma, |B_{a+b}| \leq (a+b)^\gamma) db da \\ &\leq 2 \int_0^t \int_0^t \mathbb{P}(|B_a| \leq a^\gamma, |B_{a+b} - B_a| \leq 2(a+b)^\gamma) db da \\ &= 2 \int_0^t \int_0^t \mathbb{P}(|B_a| \leq a^\gamma) \mathbb{P}(|B_b| \leq 2(a+b)^\gamma) db da \end{aligned} \quad (16)$$

Note that for any $a > 1, b > 0$, one has $\mathbb{P}(|B_a| \leq a^\gamma) \asymp a^{\gamma-\frac{1}{2}}$ and

$$\mathbb{P}(|B_b| \leq 2(a+b)^\gamma) \asymp \begin{cases} 1, & \text{if } b < a^{2\gamma}, \\ a^\gamma b^{-\frac{1}{2}}, & \text{if } a^{2\gamma} \leq b < a, \\ b^{\gamma-\frac{1}{2}}, & \text{if } b \geq a. \end{cases}$$

Applying this estimate in (16), we arrive at

$$\mathbb{E}[M(t)^2] \leq C(t^{3\gamma+\frac{1}{2}} + t^{2\gamma+1} + t^{2\gamma+1}) \leq Ct^{2\gamma+1},$$

where we used $\gamma < \frac{1}{2}$ in the second inequality. The proof of (3) is now complete.

4. Proof of Theorem 2 : dimension of all level sets

In the previous section, we have proved that almost surely, $\text{Dim}_H(\mathcal{Z}_0) = 1/2$. We are going to prove that almost surely, for every $x \in \mathbb{R}$, $\text{Dim}_H(\mathcal{Z}_x) = 1/2$.

For this, we start with the following simple lemma. For any function $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ and any interval $I \subset \mathbb{R}^+$, the oscillation of f on I is written

$$\text{Osc}_I(f) = \sup_I f - \inf_I f.$$

Lemma 5. *With probability one, there exists an integer $N \geq 1$ such that for every $n \geq N$, for any integer $\ell \in [2^{n-1}, 2^n - n^{3/2}]$, there exists at least one interval among the consecutive intervals $\{I_k := [\ell + k, \ell + k + 1]\}_{k=0, \dots, n^{3/2}-1}$ such that $\text{Osc}_{I_k}(B)$ is larger than $\log \log n$.*

Proof. Fix $n \geq 1$. For any integer interval $[k, k+1] \subset \mathcal{S}_n$, let us denote $\tilde{p}_n = \mathbb{P}(\text{Osc}_{[k, k+1]}(B) \geq \log \log n) > 0$. A standard estimate shows that $\tilde{p}_n = C_n e^{-(\log \log n)^2/2} / \log \log n$, where C_n is a constant uniformly bounded by above and below with respect to n .

The Markov property implies that the oscillation of Brownian motion on non-overlapping unit intervals are independent and identically distributed. Therefore, for any sequence of $\lfloor n^{3/2}/2 \rfloor$ consecutive intervals $I_1, \dots, I_{\lfloor n^{3/2}/2 \rfloor}$ of length 1, the probability that simultaneously the oscillation of B on each of these intervals is less than $\log \log n$ is $p_n := (1 - \tilde{p}_n)^{\lfloor n^{3/2}/2 \rfloor}$. One sees that $p_n \asymp e^{-C_n n^{3/2} e^{-(\log \log n)^2/2} / \log \log n}$, hence goes fast to 0 (C_n is another constant, still bounded away from 0 and ∞).

There are $C2^n / \lfloor n^{3/2}/2 \rfloor$ disjoint sequences of consecutive $\lfloor n^{3/2}/2 \rfloor$ integer intervals included in \mathcal{S}_n . Hence, by independence, the probability that there is at least one such sequence of $\lfloor n^{3/2}/2 \rfloor$ consecutive intervals such that the oscillation of B on all these intervals is less than $\log \log n$ is

$$\hat{p}_n := 1 - (1 - p_n)^{C2^n / \lfloor n^{3/2}/2 \rfloor}, \quad (17)$$

which tends exponentially fast to 1. The Borel-Cantelli lemma yields that there exists some (random) integer N such that for every $n \geq N$, there is always an interval I of length 1 in the $\lfloor n^{3/2}/2 \rfloor$ consecutive intervals such that the oscillation of B on I is larger than $\log \log n$.

As a conclusion, with probability one, every interval of length $n^{3/2}$ (which contains necessarily $\lfloor n^{3/2}/2 \rfloor$ consecutive intervals above) in \mathcal{S}_n contains a subinterval of length 1 on which the oscillation of B is larger than $\log \log n$. \square

We also need the following simple fact.

Lemma 6. *Let $E \subset \mathbb{R}^+$. One has*

$$\text{Dim}_{\text{H}} E = \inf \text{Dim}_{\text{H}} \left(\bigcup_{n \geq 1} \bigcup_{R \in \mathcal{R}_n} R \right),$$

where the infimum is taken over all families of intervals $(\mathcal{R}_n)_{n \geq 1}$ such that each $\mathcal{R}_n = \{R_{n,i}\}_i$ covers $E \cap \mathcal{S}_n$ with intervals of length multiple of n^2 with integer boundaries.

Proof. Set $\text{Dim}_{\text{H}} E = a$. For any covering $(\mathcal{R}_n)_{n \geq 1}$ of E , by monotonicity, $\text{Dim}_{\text{H}} \left(\bigcup_{n \geq 1} \bigcup_{R \in \mathcal{R}_n} R \right) \geq a$. To get the converse inequality, observe that the definition of the macroscopic Hausdorff dimension implies that for any $\varepsilon > 0$, there exists a covering $(\mathcal{R}_n^0)_{n \geq 1}$ of E with intervals of length multiple of n^2 in \mathcal{S}_n such that

$$\sum_{n=1}^{\infty} \sum_{R \in \mathcal{R}_n^0} \left(\frac{|R|}{2^n} \right)^{a+\varepsilon} < \infty.$$

Therefore, $\text{Dim}_{\text{H}} \left(\bigcup_{n \geq 1} \bigcup_i R_{n,i}^0 \right) \leq a + \varepsilon$. This proves the claim. \square

We now prove that the macroscopic Hausdorff dimension of Brownian level set is $1/2$ over a semi-infinite interval.

Lemma 7. *For every $x_0 \in \mathbb{R}$, almost surely, $\text{Dim}_{\text{H}} \mathcal{Z}_x \geq 1/2$ either for all $x \geq x_0$ or for all $x \leq x_0$.*

The main idea to prove this lemma is to use the oscillation property of the Brownian motion (Lemma 5): almost surely, for every large integer n , for all times $t \in \mathcal{S}_n$, the oscillation of B on $[t, t + n^{3/2}]$ is larger than $\log \log n$. In particular, when $B_t = 0$, this implies that one of the two intervals $[-(\log \log n)/2, 0]$ or $[0, (\log \log n)/2]$ is included in $B([t, t + n^{3/2}])$. Hence, either every $x \in [-(\log \log n)/2, 0]$ or every $x \in [0, (\log \log n)/2]$ can be written $B_{t'}$ for some $t' \in [t, t + n^{3/2}]$. This happens, almost surely, every time B touches 0 on \mathcal{S}_n , since the oscillation property of Lemma 5 holds for almost every trajectory, for every n . In other words, each time B touches 0, the image of B covers one large interval around 0 (either in \mathbb{R}^+ or in $\mathbb{R}^- := (-\infty, 0]$), this interval becoming larger and tending to infinity when n grows to infinity.

We prove that this implies that the macroscopic dimension of the level sets \mathcal{Z}_x are necessarily greater than the macroscopic dimension of \mathcal{Z}_0 .

Proof of Lemma 7. Let $n \geq N$ where N is the integer given in Lemma 5. We prove the Lemma for $x_0 = 0$. One knows that $\text{Dim}_{\text{H}} \mathcal{Z}_0 = 1/2$ almost surely.

Let

$$\mathcal{Q}_n = \{Q_{n,i} : i = 1, \dots, m_n\}$$

be a finite family of intervals of length multiple of n^2 inside \mathcal{S}_n . Each $Q_{n,i}$ can be split into a finite number of contiguous intervals Q of length n^2 .

Consider one of these intervals Q of length n^2 ($> n^{3/2}$). If Q intersects \mathcal{Z}_0 , by Lemma 5, there exists an interval $I \subset Q$ of length 1 on which the oscillation of B is greater than $\log \log n$. Hence, (at least) one of the two intervals $[0, \log \log n/2]$ or $[-\log \log n/2, 0]$ is necessarily included in $B(Q)$. So $Q \cap \mathcal{Z}_x \neq \emptyset$, simultaneously for all $x \in [0, \log \log n/2]$ or for all $x \in [-\log \log n/2, 0]$.

We introduce two family of sets by selecting among contiguous intervals Q of length n^2 inside $Q_{n,i}$ according to the behavior of B on these intervals:

$$\begin{aligned}\mathcal{Q}_{n,i}^+ &= \left\{ Q \subset Q_{n,i} : |Q| = n^2, Q \cap \mathcal{Z}_0 \neq \emptyset, [0, \frac{1}{2} \log \log n] \subset B(Q) \right\}, \\ \mathcal{Q}_{n,i}^- &= \left\{ Q \subset Q_{n,i} : |Q| = n^2, Q \cap \mathcal{Z}_0 \neq \emptyset, [-\frac{1}{2} \log \log n, 0] \subset B(Q) \right\}.\end{aligned}\quad (18)$$

Observe that for each $Q \subset Q_{n,i}$, only four cases may occur

- $Q \in \mathcal{Q}_{n,i}^+ \cap \mathcal{Q}_{n,i}^-$, if $B(Q) \supset [-\log \log n/2, \log \log n/2]$.
- $Q \in \mathcal{Q}_{n,i}^+ \setminus \mathcal{Q}_{n,i}^-$ if $B(Q) \supset [0, \log \log n/2]$ and $[-\log \log n/2, 0] \setminus B(Q) \neq \emptyset$.
- $Q \in \mathcal{Q}_{n,i}^- \setminus \mathcal{Q}_{n,i}^+$ if $B(Q) \supset [-\log \log n/2, 0]$ and $(0, \log \log n/2] \setminus B(Q) \neq \emptyset$.
- $Q \cap \mathcal{Z}_0 = \emptyset$.

Finally we set

$$\begin{aligned}\mathcal{Q}_n^+ &= \{Q : Q \in \mathcal{Q}_{n,i}^+ \text{ for some } i\}, \\ \mathcal{Q}_n^- &= \{Q : Q \in \mathcal{Q}_{n,i}^- \text{ for some } i\}.\end{aligned}\quad (19)$$

and

$$\mathcal{Z}^+ = \bigcup_{n \geq 1} \bigcup_{Q \in \mathcal{Q}_n^+} Q, \quad \mathcal{Z}^- = \bigcup_{n \geq 1} \bigcup_{Q \in \mathcal{Q}_n^-} Q.\quad (20)$$

Remark 3. *The construction ensures that for every $x \in [0, \log \log n/2]$, if \mathcal{R}_n is a covering of $\mathcal{Z}_x \cap \mathcal{S}_n$ by intervals I of size that are multiple of n^2 , then*

$$\bigcup_{I \in \mathcal{R}_n} (I + [-n^2, n^2]) \supset \bigcup_{Q \in \mathcal{Q}_n^+} Q.$$

The situation for $x \in [-\log \log /2, 0]$ is similar.

Now we choose $\mathcal{Q}_n = (Q_{n,i})_{i=1, \dots, m_n}$ so that it forms a covering of $\mathcal{Z}_0 \cap \mathcal{S}_n$, for every $n \geq 1$. It is obvious from the construction that

$$\mathcal{Z}_0 \subset \bigcup_{n \geq 1} \bigcup_{Q \in \mathcal{Q}_{n,i}^+ \cup \mathcal{Q}_{n,i}^-} Q = \mathcal{Z}^+ \cup \mathcal{Z}^-.$$

By the monotonicity of the macroscopic Hausdorff dimension, either $\text{Dim}_{\mathbb{H}} \mathcal{Z}^+ \geq \frac{1}{2}$, or $\text{Dim}_{\mathbb{H}} \mathcal{Z}^- \geq \frac{1}{2}$.

Assume that $\text{Dim}_H \mathcal{Z}^+ \geq \frac{1}{2}$.

Let $x > 0$, by Remark 3, for any covering $(\mathcal{R}_n)_{n \geq 1}$ of \mathcal{Z}_x , $\text{Dim}_H \mathcal{Z}^+ = \text{Dim}_H \mathcal{Z}^+ \cap [e^x, \infty) \leq \text{Dim}_H(\bigcup_n \bigcup_i R_{n,i} + [-n^2, n^2])$. It is clear that enlarging each interval $R_{n,i}$ in the covering by two intervals of length n^2 on both sides does not change the macroscopic Hausdorff dimension, as explained below.

Remark 4. Observe that by changing $R_{n,i}$ into $\tilde{R}_{n,i} := R_{n,i} + [-n^2, n^2]$, one has

$$\sum_i \left(\frac{|R_{n,i}|}{2^n} \right)^\rho \leq \sum_i \left(\frac{|\tilde{R}_{n,i}|}{2^n} \right)^\rho \leq 3^\rho \sum_i \left(\frac{|R_{n,i}|}{2^n} \right)^\rho,$$

and the nature (convergence/divergence) of the series involved in the computation of the macroscopic dimension are unchanged.

Therefore, by Lemma 6, for all $x > 0$,

$$\frac{1}{2} \leq \text{Dim}_H \mathcal{Z}^+ = \text{Dim}_H \mathcal{Z}^+ \cap [e^x, \infty) \leq \inf_n \text{Dim}_H \left(\bigcup_i R_{n,i} \right) = \text{Dim}_H \mathcal{Z}_x,$$

where the infimum is taken over all the possible coverings $(\mathcal{R}_n)_{n \geq 1}$ of \mathcal{Z}_x with intervals of length multiple of n^2 in \mathcal{S}_n .

The proof for the case $\text{Dim}_H \mathcal{Z}^- \geq \frac{1}{2}$ is similar. \square

Now we prove Theorem 2.

Proof. For the upper bound, observe that for every $x \in \mathbb{R}$, \mathcal{Z}_x is ultimately included in $E(\gamma)$ for every $0 < \gamma < \frac{1}{2}$. From this and Theorem 1 (formula (2)), we deduce that $\text{Dim}_H(\mathcal{Z}_x) \leq 1/2$.

We turn to the lower bound.

Let $(x_i)_{i \geq 1}$ be a dense sequence of real numbers. With probability one, the results of Lemma 7 apply to all the x_n 's simultaneously: almost surely for every i , either $\text{Dim}_H(\mathcal{Z}_x) \geq 1/2$ for all $x \leq x_i$ or $\text{Dim}_H(\mathcal{Z}_x) \geq 1/2$ for all $x \geq x_i$.

Now, fix $\rho \in [0, 1/2)$. Remark that the mapping $(x, \omega) \in \mathbb{R} \times \Omega \mapsto \nu_\rho^n(\mathcal{Z}_x)$ is measurable with respect to $\mathcal{B}(\mathbb{R}) \times \mathcal{F}$, where \mathcal{F} is the minimal augmented σ -algebra generated by B . This simply follows from the fact that ν_ρ^n is in fact a minimum of a finite number of terms, since the number of possible partitions of \mathcal{S}_n is finite (the intervals Q_i having integer boundaries) and the dependence on x is clearly measurable (indeed, $\mathcal{Z}_x = B^{-1}(\{x\})$).

For each positive integer i , let $X_i = \mathbf{1}_{\mathcal{A}_i}$, where \mathcal{A}_i is the event

$$\mathcal{A}_i = \left\{ \inf \left\{ \sum_{n \geq 1} \nu_\rho^n(\mathcal{Z}_x) : x \leq x_i \right\} = +\infty \right\}.$$

By the previous remark, the sequence of random variables $(X_i)_{i \geq 1}$ is measurable with respect to the tail σ -algebra generated by B . Define

$$Y = \begin{cases} \sup\{x_i : X_i = 1\}, & \text{if there exists } i \text{ such that } X_i = 1, \\ -\infty, & \text{otherwise.} \end{cases}$$

Then Y is also measurable with respect to the tail σ -algebra. The Kolmogorov 0-1 law implies that there is a deterministic number $y \in \mathbb{R} \cup \{\pm\infty\}$ such that $Y = y$ a.s. Three cases are possible:

(i) $y = +\infty$: almost surely, for all $x \in \mathbb{R}$, $\sum_{n \geq 1} \nu_\rho^n(\mathcal{Z}_x) = \infty$ which implies that $\text{Dim}_H \mathcal{Z}_x \geq \rho$ for all x .

(ii) $y = -\infty$: $X_i = 0$ for all $i \in \mathbb{N}$. Lemma 7 then gives that for all i , $\text{Dim}_H \mathcal{Z}_x \geq 1/2$ for all $x \geq x_i$, hence a contradiction since that would imply $y = +\infty$.

(iii) $y \in \mathbb{R}$: again by Lemma 7, we have :

- for all $x_i < y$ and all $x \leq x_i$, $\text{Dim}_H \mathcal{Z}_x \geq \rho$
- for all $x_i > y$, for all $x \geq x_i$, $\text{Dim}_H \mathcal{Z}_x \geq 1/2$.
- In addition, still by Lemma 7 for the fixed deterministic level y , almost surely, $\text{Dim}_H \mathcal{Z}_y \geq 1/2$.

To conclude, in any case, almost surely, $\text{Dim}_H \mathcal{Z}_x \geq \rho$ for all $x \in \mathbb{R}$.

Letting $\rho \uparrow 1/2$ concludes the proof. □

5. Concluding remark

In this article, we considered the large scale structure of certain sojourn sets where a Brownian motion visits “exceptionally” small values in a large scale dimension sense. Along the way, we use quite specific properties of the Brownian motion, for instance, the explicit probability density function of Brownian motion in Lemma 3. It would be of interest to extend the results here to general Lévy processes and more general Gaussian processes.

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