

# ON MULTIFRACTALITY AND TIME SUBORDINATION FOR CONTINUOUS FUNCTIONS

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ABSTRACT. Let  $Z : [0, 1] \rightarrow \mathbb{R}$  be a continuous function. We are interested in the existence of a decomposition of  $Z$  as  $Z = g \circ f$ , where  $g : [0, 1] \rightarrow \mathbb{R}$  is a monofractal function with exponent  $0 < H < 1$  and  $f : [0, 1] \rightarrow [0, 1]$  is a time subordinator (i.e.  $f$  is the integral of a positive Borel measure supported by  $[0, 1]$ ). We prove that such a decomposition can be found for a large class of functions. When the initial function  $Z$  is given, the monofractality exponent of the associated function  $g$  is uniquely determined. We explain why the assumptions made on  $Z$  are weak and rather natural.

We study in details classical examples of multifractal functions  $Z$  for which we exhibit the associated functions  $g$  and  $f$ . This work allows to find a very interesting relationship between self-similar functions and self-similar measures.

This provides new insights into the understanding of multifractal behaviors of functions, giving an important role to the regularity analysis of Borel measures.

## 1. INTRODUCTION AND MOTIVATIONS

Local regularity and multifractal analysis have become unavoidable issues in the past years. Indeed, physical phenomena exhibiting wild local regularity properties have been discovered in many contexts (turbulence flows, intensity of seismic waves, traffic analysis,...). From a mathematical viewpoint, the multifractal approach is also a fruitful source of interesting problems. Consequently, there is a strong need for a better theoretical understanding of the so-called multifractal behaviors. In this article, we investigate the relations between multifractal properties and time subordination for continuous functions.

The most common functions or processes used to model irregular phenomena are monofractal, in the sense that they exhibit the same local regularity at each point. Let us recall how the local regularity of a function is measured.

**Definition 1.1.** Let  $Z \in L_{loc}^\infty([0, 1])$ . For  $\alpha \geq 0$  and  $t_0 \in [0, 1]$ ,  $Z$  is said to belong to  $C_{t_0}^\alpha$  if there are a polynomial  $P$  of degree less than  $[\alpha]$  and a constant  $C$  such that, locally around  $t_0$ ,

$$(1) \quad |Z(t) - P(t - t_0)| \leq C|t - t_0|^\alpha.$$

The pointwise Hölder exponent of  $Z$  at  $t_0$  is  $h_Z(t_0) = \sup\{\alpha \geq 0 : f \in C_{t_0}^\alpha\}$ .

The singularity spectrum of  $Z$  is then defined by  $d_Z(h) = \dim\{t : h_Z(t) = h\}$  (dim stands for the Hausdorff dimension, and  $\dim \emptyset = -\infty$  by convention).

A function  $Z : [0, 1] \rightarrow \mathbb{R}$  is said to be **monofractal with exponent**  $0 < H$  when  $h_Z(t) = H$  for every  $t \in [0, 1]$ . For monofractal functions  $Z$ ,  $d_Z(H) = 1$ , while  $d_Z(h) = -\infty$  for  $h \neq H$ . Sample paths of Brownian motions or fractional Brownian motions are known to be almost surely monofractal with exponents less than 1. For reasons that appear below, **we focus on monofractal functions associated with an exponent  $H \in (0, 1]$ .**

More complex models had to be used and/or developed, for at least three reasons: the occurrence of intermittence phenomena (mainly in fluid mechanics), the presence of oscillating patterns (for instance in image processing), or the presence of discontinuities (in finance or telecommunications). Such models may have multifractal properties, in the sense that the support of their singularity spectrum is not reduced to a single point. Among these processes, whose local regularity varies badly from one point to another, we mention Mandelbrot multiplicative cascades and their extensions [6, 14, 12, 1], (generalized) multifractional Brownian motions [17, 3] and Lévy processes [4, 10] (for discontinuous phenomena).

Starting from a monofractal process as above in dimension 1, a simple and efficient way to get a more elaborate process is to compose it with a time subordinator, i.e. an increasing function or process. Mandelbrot, for instance, showed the pertinency of time subordination in the study of financial data [13]. From a theoretical viewpoint, it is also challenging to understand how the multifractal properties of a function are modified after a time change [19, 2].

A natural question is to understand the differences between the multifractal processes above and compositions of monofractal functions with multifractal subordinators.

**Definition 1.2.** *A function  $Z : [0, 1] \rightarrow \mathbb{R}$  is said to be the composition of a monofractal function with a time subordinator (CMT) when  $Z$  can be written as*

$$(2) \quad Z = g \circ f,$$

where  $g : [0, 1] \rightarrow \mathbb{R}$  is monofractal with exponent  $0 < H < 1$  and  $f : [0, 1] \rightarrow [0, 1]$  is an increasing homeomorphism of  $[0, 1]$ .

In this article, we prove that if a continuous function  $Z : [0, 1] \rightarrow \mathbb{R}$  has a "homogeneous multifractal" behavior (in a sense we define just below), then  $Z$  is CMT. Hence,  $Z$  is the composition of a monofractal function with a time subordinator, and shall simply be viewed as a complication of a monofractal model. This yields a deeper insight into the understanding of multifractal behaviors of continuous functions, and gives a more important role to the multifractal analysis of positive Borel measures (which are derivatives of time subordinators). We explain in Section 5 and 6 how this decomposition can be used to compute the singularity spectrum of the function  $Z$ .

Let us begin with two cases where a function  $Z$  is obviously CMT:

1. If  $Z$  is the integral of any positive Borel measure  $\mu$ , then  $Z = Id_{[0,1]} \circ Z$ , where the identity  $Id_{[0,1]}$  is monofractal with exponent 1 and  $Z$  is increasing.
2. Any monofractal function  $Z_H$  can be written  $Z_H = Z_H \circ Id_{[0,1]}$ , where  $Z_H$  is monofractal and  $Id_{[0,1]}$  is undoubtedly an homeomorphism of  $[0, 1]$ .

These two simple cases will be met again below.

To bring general answers to our problem and thus to exhibit another class of CMT functions, we develop an approach based on the oscillations of a function  $Z : [0, 1] \rightarrow \mathbb{R}$ . For every subinterval  $I \subset [0, 1]$ , consider the oscillations of order 1 of  $Z$  on  $I$  defined by

$$\omega_I(Z) = \sup_{t, t' \in I} |Z(t) - Z(t')| = \sup_{t \in I} Z(t) - \inf_{t \in I} Z(t).$$

**In the sequel, we assume that  $Z$  is continuous and for every non-trivial subinterval  $I$  of  $[0, 1]$ ,  $\omega_I(Z) > 0$ .** This entails that  $Z$  is nowhere locally constant, which is a natural assumption for the results we are looking for.

It is very classical that the oscillations of order 1 characterize precisely the pointwise Hölder exponents strictly less than 1 (see Section 2).

We now introduce the quantity which is the core of our construction.

For every  $j \geq 1$ ,  $k \in \{0, \dots, 2^j - 1\}$ , we consider the dyadic intervals  $I_{j,k} = [k2^{-j}, (k+1)2^{-j})$ , so that  $\bigcup_{k=0, \dots, 2^j-1} I_{j,k} = [0, 1]$ , the union being disjoint. For every  $j \geq 1$  and  $k \in \{0, \dots, 2^j - 1\}$ , for simplicity we set  $\omega_{j,k}(Z) = \omega_{I_{j,k}}(Z) (= \omega_{I_{j,k}^c}(Z))$  since  $Z$  is  $C^0$ .

**Definition 1.3.** For every  $j \geq 1$ , let  $H_j(Z)$  be the unique positive real number such that

$$(3) \quad \sum_{k=0}^{2^j-1} (\omega_{j,k}(Z))^{1/H_j(Z)} = 1.$$

We then define the monofractal exponent  $H(Z)$  of  $Z$  as

$$(4) \quad H(Z) = \liminf_{j \rightarrow +\infty} H_j(Z).$$

This quantity  $H(Z)$  characterizes the asymptotic maximal values of the oscillations of  $Z$  on the whole interval  $[0, 1]$ . This exponent is very important in our approach, because it gives an upper limit to the maximal time distortions we are allowed to apply.

It is satisfactory that  $H(Z)$  has a functional interpretation. Indeed, if  $Z$  can be decomposed as (2), then the exponent of the monofractal function  $g$  shall not depend on the oscillation approach nor on the dyadic basis. In Section 2 we explain that

$$(5) \quad \frac{1}{H(Z)} = \inf \left\{ p > 0 : Z \in B_{p,loc}^{1/p,\infty}((0,1)) \right\} = \inf \left\{ p > 0 : Z \in \mathcal{O}_p^{1/p}((0,1)) \right\},$$

where  $B_{1/q,loc}^{q,\infty}((0,1))$  and  $\mathcal{O}_p^{1/p}((0,1))$  are respectively the Besov space and *oscillation space* on the open interval  $(0,1)$  (see Jaffard in [11] for instance).

For multifractal functions  $Z$  satisfying some multifractal formalism, the exponent  $H(Z)$  can also be read on the singularity spectrum of  $Z$ . Indeed (see Section 2),  $H(Z)$  corresponds to the inverse of the largest possible slope of a straight line going through 0 and tangent to the singularity spectrum  $d_Z$  of  $Z$ .

These remarks are important to have an idea *a priori* of the monofractal exponent of  $g$  in the decomposition  $Z = g \circ f$ . They also give an intrinsic formula for  $H(Z)$ . In view of Theorem 1.7 above,  $H(Z)$  appears to be a characteristic exponent of  $Z$ , which contains *a priori* important informations regarding the local behavior of  $Z$ .

Let us come back to the two simple examples above:

1. For the integral  $Z$  of any positive measure  $\mu$ ,  $\sum_{k=0}^{2^j-1} \omega_{j,k}(Z) = \sum_{k=0}^{2^j-1} \mu(I_{j,k}) = 1$ , hence  $H_j(Z) = H(Z) = 1$ , which corresponds to the monofractal exponent of the identity  $Id_{[0,1]}$  from the oscillations viewpoint.

2. The first difficulties arise for the monofractal functions  $Z_H$ . When  $Z_H$  is monofractal of exponent  $H$ , then we don't have necessarily  $H(Z_H) = H$ . We always have  $H(Z_H) \leq 1$  (see Lemma 2.3 in Section 2), but it is always possible to construct wild counter-examples. Nevertheless, we treat in details the examples of the Weierstrass functions and the sample paths of (fractional) Brownian motions in Section 4, for which the exponent  $H(Z_H)$  meets our requirements.

Unfortunately, the knowledge of  $H(Z)$  is not sufficient to get relevant results. For instance, consider a function  $Z$  which has two different monofractal behaviors on  $[0, 1/2)$  and  $[1/2, 1]$ . Such an  $Z$  can be obtained as the continuous juxtaposition of two Weierstrass function with distinct exponents  $H_1 < H_2$ : We have  $H(Z) = H_1$ , and  $Z$  can not be

written as the composition of a monofractal function with a time subordinator. This is a consequence of Lemma 2.4, which asserts that two monofractal functions  $g_1$  and  $g_2$  of distinct exponents  $H_1$  and  $H_2$  never verify  $g_1 = g_2 \circ f$  for any continuous increasing function  $f : [0, 1] \rightarrow [0, 1]$  (indeed, such a strictly increasing function  $f$  would only have exponents smaller than 1 and thus would "dilate" time around every  $t \in [0, 1]$ , though being an homeomorphism of  $[0, 1]$ . This is impossible).

We need to introduce a homogeneity condition **C1** to get rid of these annoying and artificial cases. This condition heuristically imposes that the oscillations of any restriction of  $Z$  to a subinterval of  $[0, 1]$  have the same asymptotic properties as the oscillations of  $Z$  on  $[0, 1]$ .

**Definition 1.4. Condition C1:**

Let  $J \geq 0$ , and  $K \in \{0, \dots, 2^J - 1\}$ . Let  $Z_{J,K}$  be the function

$$Z_{J,K} : t \in [0, 1] \mapsto \frac{Z \circ \varphi_{J,K}(t)}{\omega_{J,K}(Z)} \in \mathbb{R},$$

where  $\varphi_{J,K}$  is the canonical affine contraction which maps  $[0, 1]$  to  $I_{J,K}$ .

Condition **C1** is satisfied for  $Z$  when there is a real number  $0 < H < 1$  such that for every  $J \geq 0$  and  $K \in \{0, \dots, 2^J - 1\}$ ,  $H(Z_{J,K}) = H(= H(Z))$ .

The function  $Z_{J,K}$  is a renormalized version of the restriction of  $Z$  to the interval  $I_{J,K}$ . Remark that  $H(Z_{J,K})$  does not depend on the normalization factor  $1/\omega_{J,K}(Z)$ . Although self-similar functions are good candidates to satisfy **C1**, a function  $Z$  fulfilling this condition does not need at all to enjoy such a property. In order to guarantee that  $Z$  is CMT, we strengthen the convergence toward  $H(Z_{J,K})$ .

**Definition 1.5. Condition C2:**

Assume that Condition **C1** is fulfilled. There are two positive sequences  $(\varepsilon_J)_{J \geq 0}$  and  $(\eta_J)_{J \geq 0}$  and two real numbers  $0 < \alpha < \beta$  with the following properties:

- (1)  $(\varepsilon_J)_{J \geq 0}$  and  $(\eta_J)_{J \geq 0}$  are positive non-increasing sequences that converge to zero, and

$$(6) \quad \varepsilon_J = o\left(\frac{1}{(\log J)^{2+\kappa}}\right) \text{ for some } \kappa > 0.$$

- (2) For every  $J \geq 0$  and  $K \in \{0, \dots, 2^J - 1\}$ , the sequence  $(H_j(Z_{J,K}))_{j \geq 1}$  converges to  $H = H(Z_{J,K})$  (it is not only a *liminf*, it is a *limit*) with the following convergence rate:

$$(7) \quad \text{For every } j \geq [J\eta_J], \quad |H - H_j(Z_{J,K})| \leq \varepsilon_J.$$

- (3) The following regularity condition holds:

$$(8) \quad \text{For every } j \geq [J\eta_J], \text{ for every } k \in \{0, \dots, 2^j - 1\}, \quad 2^{-j\beta} \leq \omega_{j,k}(Z_{J,K}) \leq 2^{-j\alpha}.$$

**Remark 1.6.** Part (i) is a definition.

In Part (ii), assuming that  $H(Z_{J,K})$  is a limit is of course a constraint, but not limiting in practice, since this condition holds for most of the interesting functions or (almost surely) for most of the sample paths of processes.

The convergence speed (7) is a more important constraint, but the convergence rate (6) we impose on  $(\varepsilon_J)_{J \geq 0}$  toward 0 is extremely slow, and is realized in the most common cases, as shown below.

Concerning Part (iii), the decreasing behavior (8) is not very restrictive: such a behavior is expected for a  $C^\gamma$  function.

**Theorem 1.7.** *Let  $Z : [0, 1] \rightarrow \mathbb{R}$  be a continuous function.*

*Assume that  $Z$  satisfies **C1** and **C2**.*

*Then  $Z$  is CMT and the function  $g$  in (2) is monofractal with exponent  $H(Z)$ .*

**Remark 1.8.** *Such a decomposition is of course not unique: If  $Z$  is CMT and  $w : [0, 1] \rightarrow [0, 1]$  is  $C^\infty$  and strictly increasing, then  $Z = (g \circ w) \circ (w^{-1} \circ f)$ , where  $g \circ w$  is still a monofractal function of exponent  $H(Z) < 1$  and  $w^{-1} \circ f$  is an increasing function.*

*Nevertheless, if two decompositions (2) exist respectively with functions  $g_1, g_2, f_1$  and  $f_2$ , then  $g_1$  and  $g_2$  are necessarily monofractal with the same exponent  $H(Z)$ . This is again a consequence of Lemma 2.4.*

An important consequence of Theorem 1.7 is that the (possibly) multifractal behavior of  $Z$  is essentially contained in the multifractal behavior of  $f$ . More precisely, since  $f$  is an increasing continuous function from  $[0, 1]$  to  $[0, 1]$ ,  $f$  is the integral of a positive measure, say  $\mu$ , on  $[0, 1]$ . The local regularity of  $\mu$  is classically quantified through a local dimension exponent defined for every  $t \in [0, 1]$  by

$$(9) \quad \alpha_\mu(t) = \liminf_{r \rightarrow 0^+} \frac{|\log \mu(B(t, r))|}{r}$$

where  $B(t, r)$  stands for the ball (here an interval) with center  $t$  and radius  $r$ . The singularity spectrum of  $\mu$  is then

$$(10) \quad d_\mu(\alpha) = \dim\{t : \alpha_\mu(t) = \alpha\}.$$

It is very easy to see that if  $Z$  is CMT with a monofractal function with exponent  $0 < H < 1$  and  $\alpha_\mu(t_0) = \alpha$ , then  $h_Z(t_0) \geq \alpha H$ , but equality does not hold true in general. Nevertheless, for measures  $\mu$  satisfying a multifractal formalism, for every  $h \geq 0$ ,  $d_f(h) = d_\mu(h/H)$ . This point is justified in Section 7. In this case there is a direct relationship between the singularity spectrum of  $Z$  and the one of  $\mu$ . This applies in particular to the self-similar functions below.

As a conclusion, Theorem 1.7 increases the role of the multifractal analysis of measures, since for the functions satisfying **C1** and **C2**, their multifractal behavior is mainly ruled by the behavior of  $\mu$ .

As an application of Theorem 1.7, we prove Theorem 1.10, which relates the so-called self-similar functions  $Z$  introduced in [9] with the self-similar measures naturally associated with the similitudes defining  $Z$ .

Let us recall the definition of self-similar functions. Let  $\phi$  be a Lipschitz function on  $[0, 1]$  (we suppose that the Lipschitz constant  $C_\phi$  equals 1, without loss of generality), and let  $S_0, S_1, \dots, S_{d-1}$  be  $d$  contractive similitudes satisfying:

- (1) for every  $i \neq j$ ,  $S_i((0, 1)) \cap S_j((0, 1)) = \emptyset$  (open set condition),
- (2)  $\bigcup_{i=0}^{d-1} S_i([0, 1]) = [0, 1]$  (the intervals  $S_i([0, 1])$  form a covering of  $[0, 1]$ ).

We denote by  $0 < r_0, r_1, \dots, r_{d-1} < 1$  the ratios of the non-trivial similitudes  $S_0, \dots, S_{d-1}$ .

By construction  $\sum_{k=0}^{d-1} r_k = 1$ . Let  $\lambda_0, \lambda_1, \dots, \lambda_{d-1}$  be  $d$  non-zero real numbers, which satisfy

$$(11) \quad 0 < \chi_{\min} = \min_{k=0, \dots, d-1} \left| \frac{r_k}{\lambda_k} \right| \leq \chi_{\max} = \max_{k=0, \dots, d-1} \left| \frac{r_k}{\lambda_k} \right| < 1.$$

**Definition 1.9.** A function  $Z : [0, 1] \rightarrow [0, 1]$  is called *self-similar* when  $Z$  satisfies the following functional equation

$$(12) \quad \forall t \in [0, 1], \quad Z(t) = \sum_{k=0}^{d-1} \lambda_k \cdot (Z \circ (S_k)^{-1})(t) + \phi(t).$$

Relation (11) ensures that  $Z$  exists and is unique [9].

Let us consider the unique exponent  $\beta > 1$  such that

$$(13) \quad \sum_{k=0}^{d-1} |\lambda_k|^\beta = 1.$$

This  $\beta$  is indeed greater than 1, since  $\sum_{k=0}^{d-1} r_k = 1$  and  $|\lambda_k| > r_k$  for all  $k$  by (11). With the probability vector  $(p_0, p_1, \dots, p_{d-1}) = (|\lambda_0|^\beta, |\lambda_1|^\beta, \dots, |\lambda_{d-1}|^\beta)$  and the similitudes  $(S_k)_{k=0, \dots, d-1}$  can be associated the unique self-similar probability measure  $\mu$  satisfying

$$(14) \quad \mu = \sum_{k=0}^{d-1} p_k \cdot (\mu \circ S_k^{-1}).$$

**Theorem 1.10.** Let  $Z$  be defined by (12). Let  $\kappa = \frac{\chi_{\max}}{1 - \chi_{\max}}$ .

Then, either  $Z$  is a  $\kappa$ -Lipschitz function, or  $Z$  satisfies **C1** and **C2**, and is CMT.

Moreover, there is a monofractal function  $g$  of exponent  $1/\beta$  such that

$$(15) \quad \text{for every } t \in [0, 1], \quad Z(t) = g(\mu[0, t]),$$

where  $\mu$  is the self-similar measure (14) naturally associated with the parameters used to define  $Z$ . In particular, for every  $h \geq 0$ ,  $d_Z(h) = d_\mu(\beta h)$ .

Theorem 1.10 establishes a very satisfactory relationship between self-similar functions and self-similar measures, that somehow was expected. It also asserts that the choice of the function  $\phi$  influences only the monofractal function  $g$ , not the time subordinator. Since the self-similar measure  $\mu$  verifies the multifractal formalism (see [6]), the multifractal analysis of  $Z$  follows from that of  $\mu$ . This also illustrates our purpose: in this case, the local regularity analysis of the functions we consider reduces to the local regularity analysis of some Borel measure  $\mu$ .

The paper is organized as follows. Section 2 contains preliminary results on oscillations of functions, the explanation of the characterization of  $H(Z)$  (5), and the possible extensions of Theorem 1.7 and a short discussion for exponents greater than 1. In Section 3, Theorem 1.7 is proved, by explicitly constructing the monofractal function  $g$  and the time subordinator  $f$ . In Section 4, 5 and 6, we detail several classes of examples to which Theorem 1.7 applies. First we prove that the usual monofractal functions  $Z$  with exponents  $H$  verify **C1** and **C2**. We prove Theorem 1.10 in Section 5. We explicitly compute and plot the time subordinator and the monofractal function for a classical family of multifractal functions  $(Z_a)_{a \in [0, 1]}$  which include Bourbaki's and Perkin's functions. Finally we

quickly explain the transfer of multifractality properties from  $f$  to  $Z$  when the measure  $\mu$  associated with  $f$  fulfills a multifractal formalism.

Let us finish by the direct by-products and the possible extensions of this work:

The reader can check that the proof below can be adapted to more general contexts:

- the dyadic basis can be replaced by any  $b$ -adic basis.
- if  $(\varepsilon_J)$  converges to zero (without any given convergence rate like (6)), then (under slight modifications of  $(\eta_J)$ ) the same result holds true. We focused on a simpler case, but in practice, a convergence rate  $\varepsilon_J = o\left(\frac{1}{(\log J)^{2+\kappa}}\right)$  shall always be always obtained.
- The fact the the quantities  $H(Z_{J,K})$  are limits is only used at the beginning of the proof. In fact, only the existence of the scale  $[J\eta_J]$  such that (7) and (8) hold true at scale  $[J\eta_J]$  is determinant. In particular, the conditions may be relaxed, without adding any substantial theoretical improvement to our result.

## 2. PRELIMINARY RESULTS

**2.1. Oscillations and pointwise regularity.** Let us recall the characterization of the pointwise Hölder exponents smaller than 1 in terms of oscillations of order 1 (see for instance Jaffard in [11]).

**Lemma 2.1.** *Let  $Z : [0, 1] \rightarrow \mathbb{R}$  a  $C^\gamma$  function, for some  $\gamma > 0$ . Assume that  $h_Z(t) < 1$ . Then*

$$h_Z(t) = \liminf_{r \rightarrow 0^+} \frac{|\log \omega_{B(t,r)}(Z)|}{|\log r|} = \liminf_{j \rightarrow +\infty} \frac{|\log_2 \omega_{B(t,2^{-j})}(Z)|}{j}.$$

In Lemma 2.2, we impose some uniform behavior of the oscillations of  $Z$  on a nested sequence of coverings of  $[0, 1]$ . This is used later to prove the monofractality property of the function  $g$  in the decomposition  $Z = g \circ f$  (Section 3), and also to decompose self-similar functions (in Section 5).

**Lemma 2.2.** *Let  $Z : [0, 1] \rightarrow \mathbb{R}$  be a continuous function,  $t \in (0, 1)$  and  $H \in (0, 1)$ .*

*Suppose that there exists an infinite sequence  $(T_n)$  of coverings of  $[0, 1]$  such that*

- *each  $T_n$  is a finite sequence of disjoint non-trivial intervals of  $[0, 1]$ , such that  $\bigcup_{T \in T_n} T = [0, 1]$ ,*
- *$\lim_{n \rightarrow +\infty} \max_{T \in T_n} |T| = 0$ ,*
- *each interval  $T$  in  $T_n$  is contained in a unique interval  $T'$  of  $T_{n-1}$ ,*
- *for every  $T \in T_n$  and  $T \subset T' \in T_{n-1}$ , we have  $|T'|^{1+\xi_n} \leq |T| \leq |T'|$ , for some positive sequence  $(\xi_n)_{n \geq 1}$  that converges to zero when  $n \rightarrow +\infty$ .*

*Then:*

- (1) *If there exists a positive sequence  $(\kappa_n)_{n \geq 1}$  such that for every  $T \in T_n$ ,  $\omega_T(Z) \leq |T|^{H-\kappa_n}$ , then for every  $t \in [0, 1]$ ,  $h_Z(t) \geq H$ .*
- (2) *If there exists a positive sequence  $(\kappa_n)_{n \geq 1}$  such that for every  $T \in T_n$ ,  $\omega_T(Z) \geq |T|^{H+\kappa_n}$ , then for every  $t \in [0, 1]$ ,  $h_Z(t) \leq H$ .*

Remark that in part (2) of this Lemma, the property needs to be satisfied only for a subsequence  $(n_k)_{k \geq 1}$  of integers.

*Proof.* Let  $t \in (0, 1)$ , and  $r > 0$  small enough. For every  $n \geq 1$ ,  $t$  belongs to one interval  $T \in T_n$ , that we denote  $T_n(t)$ . Denote by  $n_r$  the smallest integer  $n$  so that  $T_n(t) \subset B(t, r)$ . By construction,  $t \in T_{n_r-1}(t)$  and  $|T_{n_r-1}(t)| \geq r$  (since  $T_{n_r-1}(t) \not\subset B(t, r)$ ). By

the fourth property of the sequence  $(T_n)$ , we have  $2r \geq |T_{n_r}(t)| \geq |T_{n_r-1}(t)|^{1+\xi_{n_r}} \geq r^{1+\xi_{n_r}}$ .

Let us start by part (2), which is very easy to get. We have

$$\omega_{B(t,r)}(Z) \geq \omega_{T_{n_r}(t)}(Z) \geq |T_{n_r}(t)|^{H+\kappa_{n_r}} \geq r^{(1+\xi_{n_r})(H+\kappa_{n_r})}.$$

Applying Lemma 2.1, and using that the two sequences  $\xi_{n_r}$  and  $\kappa_{n_r}$  go to zero when  $r$  goes to zero, we obtain  $h_Z(t) \leq H$ .

We now focus on part (1), which is slightly more delicate. If  $B(t,r) \subset T_{n_r-1}(t)$ , then we have  $\omega_{B(t,r)}(Z) \leq \omega_{T_{n_r-1}(t)}(Z) \leq |T_{n_r-1}(t)|^{H-\kappa_{n_r-1}} \leq (2r)^{(H-\kappa_{n_r-1})/(1+\xi_{n_r-1})}$ .

If  $B(t,r) \not\subset T_{n_r-1}(t)$ , then there is an integer  $p$  (which depends on  $r$ ) such that  $B(t,r) \setminus T_{n_r-1}(t)$  is covered by one interval  $T \in T_p$  and not covered by any interval of  $T_{p+1}$ . Using the same arguments as above, we get  $|T| \leq |B(t,r) \setminus T_{n_r-1}(t)|^{1/(1+\xi_{p+1})} \leq r^{1/(1+\xi_{p+1})}$  (remark that  $|B(t,r) \setminus T_{n_r-1}(t)| \leq r$ ).

Now we have

$$\begin{aligned} \omega_{B(t,r)}(Z) &\leq \omega_{T_{n_r-1}(t)}(Z) + \omega_{B(t,r) \setminus T_{n_r-1}(t)}(Z) \\ &\leq (2r)^{(H-\kappa_{n_r-1})/(1+\xi_{n_r-1})} + |T|^{H-\kappa_p} \\ &\leq (2r)^{(H-\kappa_{n_r-1})/(1+\xi_{n_r-1})} + r^{(H-\kappa_p)/(1+\xi_{p+1})} \end{aligned}$$

Since  $\kappa_{n_r}$ ,  $\xi_{n_r}$ ,  $\kappa_p$  and  $\xi_p$  converge to 0 as  $r \rightarrow 0$ , Lemma 2.1 yields  $h_Z(t) \geq H$ .  $\square$

**2.2. Two easy properties for the study of  $H(Z)$ .** Let us begin with an easy upper-bound for  $H(Z)$ , which explains why we focus only on monofractal function with exponents less than 1.

**Lemma 2.3.** *Let  $Z : [0, 1] \rightarrow \mathbb{R}$  be a non-constant continuous function. Then  $H(Z) \leq 1$ .*

*Proof.* We can assume without loss of generality that  $\omega_{[0,1]}(Z) = 1$ . Let  $j \geq 1$ . By construction,  $\sum_{k=0}^{2^j-1} \omega_{j,k}(Z) \geq 1$ . In order to have (3), we necessarily have  $H_j(Z) \leq 1$ . Hence the result.  $\square$

The following Lemma implies that for a CMT function  $Z$ , the associated monofractal exponent is necessarily unique.

**Lemma 2.4.** *Let  $g_1$  and  $g_2$  be two real monofractal functions on  $[0, 1]$  of distinct exponents  $0 < H_1 < H_2 < 1$ . There is no continuous strictly increasing function  $f : [0, 1] \rightarrow [0, 1]$  such that  $g_1 = g_2 \circ f$ .*

Indeed, if  $Z$  is CMT and can be written  $Z = g_1 \circ f_1 = g_2 \circ f_2$ , then  $g_1 = g_2 \circ (f_2 \circ f_1^{-1})$ , and thus  $g_1$  and  $g_2$  have the same monofractality exponents.

*Proof.* Suppose that such a function  $f$  exists. Let  $\varepsilon > 0$ . This function  $f$  is Lebesgue-almost everywhere differentiable. There is a set  $E$  of positive Lebesgue measure such that for every  $t \in E$ ,  $f'(t) > 0$ . Around such a  $t$ , we have  $f(t+h) - f(t) = f'(t) \cdot h + o(t)$ . Consequently, since  $h_{g_2}(f(t)) = H_2$ , for every  $|h|$  small enough we have

$$|(g_2 \circ f)(t+h) - (g_2 \circ f)(t)| \leq |f(t+h) - f(t)|^{H_2-\varepsilon} \leq C|h|^{H_2-\varepsilon}.$$

This shows that  $h_{g_2 \circ f}(t) \geq H_2$ . Using again that  $h_{g_2}(f(t)) = H_2$ , there is a sequence  $(h'_n)_{n \geq 1}$  converging to zero such that for every  $n \geq 1$ ,  $|g_2(f(t) + h'_n) - g_2(f(t))| \geq |h'_n|^{H_2+\varepsilon}$ . Choosing  $h_n$  so that  $f(t+h_n) = f(t) + h'_n$ , we see that

$$|(g_2 \circ f)(t+h_n) - (g_2 \circ f)(t)| \geq |f(t+h_n) - f(t)|^{H_2+\varepsilon} \geq C|h_n|^{H_2+\varepsilon}.$$



This holds for an infinite number of real numbers  $(h_n)$  converging to zero. Hence  $h_{g_2 \circ f}(t) = H_2$ , which contradicts  $h_{g_1}(t) = H_1$ .  $\square$

**2.3. A functional interpretation of  $H(Z)$ .** Note first that the previous results hold in the case where a  $b$ -adic basis,  $b \geq 2$ , is used instead of the dyadic basis. In fact, there is a functional interpretation of the exponent  $H(Z)$ , independent of any basis, provided by the *Oscillation spaces* of Jaffard [11] and the Besov spaces. Let us recall their definition, that we adapt to our context of nowhere differentiable functions.

Let  $Z$  be a  $C^\gamma$  function on  $(0, 1)$ , where  $C^\gamma$  is the global homogeneous Hölder space and  $\gamma > 0$ . Since [9] where the theoretical foundations of multifractal analysis of functions were given, a quantity classically considered when performing the multifractal analysis of  $Z$  is the scaling function  $\eta_Z(p) = \sup \left\{ s > 0 : Z \in B_{p,loc}^{s/p,\infty}((0, 1)) \right\}$ .

Later, in [11], Jaffard also proved the pertinency in multifractal analysis of his oscillation spaces  $\mathcal{O}_p^{s/p}((0, 1))$ , whose definitions are based on wavelet leaders (we do not need much more details here). He also considered the associated scaling function  $\zeta_Z(p) = \sup \left\{ s > 0 : Z \in \mathcal{O}_p^{s/p}((0, 1)) \right\}$ .

Finally, still in [11], Jaffard studied the spaces  $\mathcal{V}_p^{s/p}((0, 1))$ , which are closely related to our exponent  $H(Z)$ , defined as follows: Denote, for every integer  $j \geq 1$  and every  $k \in \{0, \dots, 2^j - 1\}$ ,  $\Omega_{j,k}(Z) = \omega_{[k2^{-j}-32^{-j}, k2^{-j}+32^{-j}]}(Z)$ , and consider the associated scaling function

$$\nu_Z(p) = 1 + \liminf_{j \rightarrow +\infty} \frac{1}{-j} \log_2 \sum_{k=0}^{2^j-1} (\Omega_{j,k}(Z))^p.$$

For  $p > 0$  fixed, it is obvious that there is a constant  $C_p > 1$  such that

$$1/C_p \sum_{k=0}^{2^j-1} (\omega_{j,k}(Z))^p \leq \sum_{k=0}^{2^j-1} (\Omega_{j,k}(Z))^p \leq C_p \sum_{k=0}^{2^j-1} (\omega_{j,k}(Z))^p,$$

since  $\omega_{j,k}(Z) \leq \Omega_{j,k}(Z) \leq \sum_{l \in \{-3, -2, \dots, 2, 3\}} \omega_{j,k+l}(Z)$ . Subsequently,

$$\nu_Z(p) = 1 + \liminf_{j \rightarrow +\infty} \frac{1}{-j} \log_2 \sum_{k=0}^{2^j-1} (\omega_{j,k}(Z))^p.$$

Comparing the definition of  $H(Z)$  with this formula, we easily see that  $H(Z)$  is the unique positive real number such that  $\nu_Z(1/H(Z)) = 1$ .

The main point is that the three scaling functions  $\eta_Z$ ,  $\zeta_Z$  and  $\nu_Z$  coincide as soon as  $p \geq 1$  [11], and  $\eta_Z(1/H(Z)) = \zeta_Z(1/H(Z)) = 1$ . Using the property of the Besov domains, we have

$$H(Z)^{-1} = \inf \left\{ p > 0 : Z \in B_{p,loc}^{1/p,\infty}((0, 1)) \right\} = \inf \left\{ p > 0 : Z \in \mathcal{O}_p^{1/p}((0, 1)) \right\}.$$

**2.4. Precisions for functions satisfying a multifractal formalism.** Consider the scaling function  $\zeta_Z(p)$  above. Then for any function  $Z$  having some global Hölder regularity [11],  $Z$  is said to obey the multifractal formalism for functions if its singularity spectrum is obtained as the Legendre transform of its scaling function, i.e.

$$\text{for every } h \geq 0, \quad d_Z(h) = \inf_{p \in \mathbb{R}} (ph - \zeta_Z(p) + 1) \quad (\in \mathbb{R}^+ \cup \{-\infty\}).$$

In particular, since  $\zeta_Z(1/H(Z)) = 1$ , we always have  $d_Z(h) \leq h/H(Z)$  (by using  $p = 1/H(Z)$  in the inequality above).

Moreover, assume that  $h_c = \zeta'_Z(1/H(Z))$  exists and that  $Z$  satisfies the multifractal formalism associated with  $\zeta_Z$  at the exponent  $h_c$ . This means that the inequality above holds true for  $h = h_c$ , i.e.  $d_Z(h_c) = h_c/H(Z)$ .

From the two last properties we get that  $1/H(Z)$  is the slope of the tangent to the (concave hull of the) singularity spectrum of  $Z$ , as claimed in the introduction.

**2.5. Possible extensions for monofractal functions exponents greater than 1.** Let us finally say a few words about functions having regularity exponents greater than 1. The presence of a polynomial in the definition (1) of the pointwise Hölder exponent is a source of problems when analyzing the local regularity after time subordination. Indeed, suppose that a continuous function  $f_1$  behaves like  $|t - t_0|^\alpha$  ( $0 < \alpha < 1$ ) around a point  $t_0$ , and that another continuous function  $g_1$  behaves like  $a(t - t_0) + |t - t_0|^{3/2}$  ( $a \neq 0$ ) around  $t_0 = f_1(t_0)$ . Then  $h_{f_1}(t_0) = \alpha$ ,  $h_{g_1}(t_0) = 3/2$ , but  $h_{g_1 \circ f_1}(t_0) = \alpha$ , which is different than the expected regularity  $3\alpha/2$ . Applying the construction of Section 3 below and getting a decomposition of a function  $Z$  as  $Z = g \circ f$ , because of such problems, we didn't find any way to guarantee the monofractality of  $g$ .

This is related to the fact that, still for the just above toy example,  $\omega_{B(t_0,r)}(g_2) \sim 2ar$  when  $r$  is small enough, while one would expect  $\omega_{B(t_0,r)}(g_2) \sim r^{3/2}$ . The use of oscillations of order greater than 2 (so that  $\omega_{B(t_0,r)}^2(g_2) \sim r^{3/2}$ ) was not sufficient for us to prove Theorem 1.7 for functions with monofractal exponent greater than 1.

As a consequence, the problem where the "theoretical" monofractal function  $g$  in the decomposition  $Z = g \circ f$  has exponents greater than 1 is open and of interest.

### 3. THE EXISTENCE OF A CMT DECOMPOSITION: THEOREM 1.7

Assume that conditions **C1** and **C2** are fulfilled.

The functions  $g$  and  $F$  are constructed iteratively.

First remark that since  $(\eta_j)$  converges to zero, one can also assume, by first replacing  $\eta_j$  by  $\max(\eta_j, 1/\log j)$  and then by imposing that  $(\eta_j)$  is non-increasing, that the sequence  $(\eta_j)$  satisfies:

- for every  $j \geq 1$ ,  $\eta_j \geq 1/\log j$ ,
- $(j\eta_j)$  is now a non-decreasing sequence and  $j\eta_j \rightarrow +\infty$  when  $j \rightarrow +\infty$ ,
- $(\eta_j)$  still satisfies (7) and (8).

**3.1. First step of the construction of  $g$  and  $f$ .** The exponent  $H(Z_{0,0}) = H(Z) = H$  is the limit of the sequence  $H_j(Z)$ , so there exists a generation  $J_0 \geq 1$  such that for every  $j \geq J_0$ ,  $|H - H_j(Z)| \leq \varepsilon_0$ .

We set  $H_0 = H_{J_0}(Z)$ , and by construction we have  $\sum_{k=0}^{2^{J_0}-1} (\omega_{J_0,k}(Z))^{1/H_0} = 1$ .

We then define the first step of the construction of the function  $f$ : we set

$$f_0(t) = \sum_{k'=0}^{k-1} (\omega_{J_0,k'}(Z))^{1/H_0} + (\omega_{J_0,k}(Z))^{1/H_0} (2^{J_0}t - k) \text{ if } t \in I_{J_0,k}.$$

This function  $f_0$  is strictly increasing, continuous and affine on each dyadic interval. Moreover,  $f_0([0, 1]) = [0, 1]$ . Let us denote  $U_{J_0,k}$  the image of the interval  $I_{J_0,k}$  by  $f_0$ , for every  $k \in \{0, \dots, 2^{J_0} - 1\}$ . The set of intervals  $\{U_{J_0,k} : k \in \{0, \dots, 2^{J_0} - 1\}\}$  clearly forms a partition of  $[0, 1]$ . One remarks that

$$(16) \quad \forall k \in \{0, \dots, 2^{J_0} - 1\}, \quad |f_0(I_{J_0,k})| = |U_{J_0,k}| = (\omega_{J_0,k}(Z))^{1/H_0}.$$

The first step of the construction of  $g$  is then naturally achieved as follows: we set

$$\begin{aligned} g_0(y) &= Z((f_0)^{-1}(y)) \text{ for } y \in [0, 1], \\ \text{or equivalently } g_0(f_0(t)) &= Z(t) \text{ for } t \in [0, 1]. \end{aligned}$$

This function  $g_0$  maps any interval  $U_{J_0,k}$  to the interval  $Z(I_{J_0,k})$ , and thus satisfies:

$$\omega_{U_{J_0,k}}(g_0) = \omega_{J_0,k}(Z) = |U_{J_0,k}|^{H_0}.$$

As a last remark, there are two real numbers  $0 < \alpha' < \beta'$  such that for every  $k$   $2^{-J_0\beta'/H_0} \leq |U_{J_0,k}| \leq 2^{-J_0\alpha'/H_0}$ . Without loss of generality, we can assume that  $\alpha' = \alpha$  and  $\beta' = \beta$  ( $\alpha$  and  $\beta$  appear in condition **C2**) by changing  $\alpha$  into  $\min(\alpha', \alpha)$  and  $\beta = \max(\beta, \beta')$ , so that

$$(17) \quad \text{for every } k, \quad 2^{-J_0\beta/H_0} \leq |U_{J_0,k}| \leq 2^{-J_0\alpha/H_0}.$$

**3.2. First iteration to get the second step of the construction of  $g$  and  $f$ .** We perform the second step of the construction. Let us focus on one interval  $I_{J_0,K}$ , on which we refine the behavior of  $f_0$ . By condition **C2** and especially (7), we have

$$(18) \quad \sum_{k'=0, \dots, 2^{[J_0\eta_{J_0}] - 1}} (\omega_{[J_0\eta_{J_0}], k'}(Z_{J_0,K}))^{1/H_1} = 1,$$

where  $H_1 = H_{[J_0\eta_{J_0}]}(Z_{J_0,K})$  satisfies  $|H - H_1| \leq \varepsilon_{J_0}$ .

Let  $J_1 = J_0 + [J_0\eta_{J_0}]$ , hence  $J_1 - J_0 = [J_0\eta_{J_0}]$ . Remark that, by (8), we have for every  $k' \in \{0, \dots, 2^{[J_0\eta_{J_0}] - 1}\}$

$$(19) \quad 2^{-[J_0\eta_{J_0}]\beta} \leq |\omega_{[J_0\eta_{J_0}], k'}(Z_{J_0,K})| \leq 2^{-[J_0\eta_{J_0}]\alpha}.$$

Now, remembering the definition of  $Z_{J_0,K}$ , we obtain that for every  $k' \in \{0, \dots, 2^{J_1 - J_0} - 1\}$ ,

$$(20) \quad \omega_{J_1 - J_0, k'}(Z_{J_0,K}) = \frac{\omega_{J_1, K2^{J_1 - J_0} + k'}(Z)}{\omega_{J_0, K}(Z)}.$$

Consequently, (18) is equivalent to

$$\sum_{k=0, \dots, 2^{J_1 - 1} : I_{J_1, k} \subset I_{J_0, K}} (\omega_{J_1, k}(Z))^{1/H_1} = (\omega_{J_0, K}(Z))^{1/H_1},$$

and thus

$$\sum_{k=0, \dots, 2^{J_1 - 1} : I_{J_1, k} \subset I_{J_0, K}} (\omega_{J_1, k}(Z))^{1/H_1} (\omega_{J_0, K}(Z))^{1/H_0 - 1/H_1} = (\omega_{J_0, K}(Z))^{1/H_0}.$$

We now define the function  $f_1$  as a refinement on  $f_0$  on the dyadic interval  $I_{J_0, K}$ . We set for every  $k \in \{K2^{J_1 - J_0}, \dots, (K+1)2^{J_1 - J_0} - 1\}$  and for  $t \in I_{J_1, k}$

$$\begin{aligned} f_1(t) &= f_0(K2^{-J_0}) \\ &+ \sum_{k'=K2^{J_1 - J_0}}^{k-1} (\omega_{J_1, k'}(Z))^{1/H_1} (\omega_{J_0, K}(Z))^{1/H_0 - 1/H_1} \\ &+ (\omega_{J_1, k}(Z))^{1/H_1} (\omega_{J_0, K}(Z))^{1/H_0 - 1/H_1} (2^{J_1}t - k). \end{aligned}$$

This can be achieved simultaneously on every dyadic interval  $I_{J_0, K}$ ,  $K \in \{0, \dots, 2^{J_0} - 1\}$ , by using the same generation  $J_1$  for the subdivision (indeed, condition **C2** ensures that the convergence rate of  $H_j(Z_{J_0, k})$  does not depend on  $k$ ). The obtained function is again an increasing continuous function, affine on every dyadic interval of generation  $J_1$ .

Let us denote  $U_{J_1,k}$  the image of the interval  $I_{J_1,k}$  by  $f_1$ , for every  $k \in \{0, \dots, 2^{J_1} - 1\}$ . The set of intervals  $\{U_{J_1,k} : k \in \{0, \dots, 2^{J_1} - 1\}\}$  again forms a partition of  $[0, 1)$ . We get

$$(21) \quad \forall k \in \{0, \dots, 2^{J_1} - 1\}, \quad |U_{J_1,k}| = (\omega_{J_1,k}(Z))^{1/H_1} (\omega_{J_0,K}(Z))^{1/H_0 - 1/H_1}.$$

but the main point is that we did not change the size of the oscillations of  $f_0$  on the dyadic intervals of generation  $J_0$ , i.e.  $f_1(I_{J_0,K}) = f_0(I_{J_0,K})$ .

The second step of the construction of  $g$  is realized by refining the behavior of  $g_0$ : Set

$$g_1(y) = Z((f_1)^{-1}(y)) \text{ for } y \in [0, 1].$$

This function  $g_1$  maps any interval  $U_{J_1,k}$  to the interval  $Z(I_{J_1,k})$ , and thus satisfies:

$$\omega_{U_{J_1,k}}(g_1) = \omega_{J_1,k}(Z) \text{ with } |U_{J_1,k}| = (\omega_{J_1,k}(Z))^{1/H_1} (\omega_{J_0,K}(Z))^{1/H_0 - 1/H_1}.$$

Finally, we want to compare the size of the interval  $U_{J_1,k}$  with the size of its father interval (in the preceding generation)  $U_{J_0,K}$ . For this, let us choose  $k \in \{0, \dots, 2^{J_1} - 1\}$  and  $K \in \{0, \dots, 2^{J_0} - 1\}$  are such that  $I_{J_1,k} \subset I_{J_0,K}$  (hence  $k$  can be written  $k = K \cdot 2^{J_1 - J_0} + k'$  with  $k' \in \{0, \dots, 2^{J_1 - J_0} - 1\}$ ). Then, by (20),

$$\begin{aligned} |U_{J_1,k}| &= (\omega_{J_0,K}(Z))^{1/H_0} (\omega_{J_1 - J_0, k'}(Z_{J_0,K}))^{1/H_1} \\ &= |U_{J_0,K}| (\omega_{J_1 - J_0, k'}(Z_{J_0,K}))^{1/H_1}. \end{aligned}$$

Using (19) we get

$$|U_{J_1,k}| \geq |U_{J_0,K}| 2^{-(J_1 - J_0)\beta/H_1} = |U_{J_0,K}| 2^{-[J_0\eta_{J_0}]\beta/H_1}.$$

On the other side, we know by (17) that  $|U_{J_0,k}| \leq 2^{-J_0\alpha/H_0}$ , hence

$$|U_{J_0,k}| \geq |U_{J_1,k}| \geq |U_{J_0,k}|^{1 + \eta_{J_0} \frac{\beta H_0}{\alpha H_1}},$$

where the left inequality simply comes from the fact that  $I_{J_1,k} \subset I_{J_0,K}$ .

**3.3. General iterating construction of  $g$  and  $f$ .** This procedure can be iterated. Assume that the sequences  $(J_p)_{p \geq 1}$ ,  $(f_p)_{p \geq 1}$  and  $(g_p)_{p \geq 1}$  are constructed for every  $p \leq n$ , and that they satisfy:

- (1) for every  $1 \leq p \leq n$ ,  $J_p = J_{p-1} + [J_{p-1}\eta_{J_{p-1}}]$  and  $|H - H_p| \leq \varepsilon_{J_{p-1}}$ ,
- (2) for every  $1 \leq p \leq n$ ,  $f_p$  is a continuous strictly increasing function, affine on each dyadic interval  $I_{J_p,k}$  and if we set  $f_p(I_{J_p,k}) = U_{J_p,k}$ , then

$$(22) \quad |f_p(I_{J_p,k})| = |U_{J_p,k}| = (\omega_{J_p,k}(Z))^{1/H_p} \prod_{m=0}^{p-1} (\omega_{J_m, K_m(k)}(Z))^{1/H_m - 1/H_{m+1}},$$

where  $K_m(k)$  is the unique integer such that  $I_{J_p,k} \subset I_{J_m, K_m(k)}$ , for  $m < p$ ,

- (3) For every  $1 \leq p \leq n$ , the set of intervals  $\{U_{J_p,k} : k \in \{0, \dots, 2^{J_p} - 1\}\}$  forms a partition of  $[0, 1)$ .
- (4) For every  $1 \leq p \leq n$ , if  $U_{J_n,k} \subset U_{J_{p-1}, K_{p-1}(k)}$ , then

$$|U_{J_{p-1}, K_{p-1}(k)}|^{1 + \eta_{J_{p-1}} \frac{\beta H_{p-1}}{\alpha H_p}} \leq |U_{J_p,k}| \leq |U_{J_{p-1}, K_{p-1}(k)}|,$$

- (5) for every  $1 \leq p \leq n$ , for  $y \in [0, 1]$ ,  $g_p(y) = Z((f_p)^{-1}(y))$
- (6) for every  $1 \leq p \leq n$ , for every  $k \in \{0, \dots, 2^p - 1\}$ , we have  $f_m(k2^{-p}) = f_p(k2^{-p})$  for every  $p \leq m \leq n$ .

The last item ensures that once the value of  $f_p$  at  $k2^{-p}$  has been chosen, every  $f_m$ ,  $m \geq p$ , will take the same value at  $k2^{-p}$ .

To build  $f_{n+1}$  and  $g_{n+1}$ , the procedure is the same as above. We use  $J_{n+1} = J_n + [J_n \eta_{J_n}]$ , and we focus on one interval  $I_{J_n, K}$ . We have by (7)

$$\sum_{k=0, \dots, 2^{J_{n+1}-J_n-1}} (\omega_{J_{n+1}-J_n, k}(Z_{J_n, K}))^{1/H_{n+1}} = (\omega_{J_n, K}(Z))^{1/H_{n+1}},$$

where  $H_{n+1} = H_{[J_n \eta_{J_n}]}(Z_{J_n, K})$  satisfies  $|H - H_{n+1}| \leq \varepsilon_{J_n}$ . we have for every  $k' \in \{0, \dots, 2^{[J_n \eta_{J_n}]} - 1\}$

$$(23) \quad 2^{-[J_n \eta_{J_n}] \beta} \leq |\omega_{[J_n \eta_{J_n}], k'}(Z_{J_n, K})| \leq 2^{-[J_n \eta_{J_n}] \alpha}.$$

and

$$(24) \quad \omega_{J_{n+1}-J_n, k'}(Z_{J_n, K}) = \frac{\omega_{J_{n+1}, K2^{J_{n+1}-J_n+k'}}(Z)}{\omega_{J_n, K}(Z)}.$$

The same manipulations as above yield

$$(25) \quad \sum_{k=0, \dots, 2^{J_{n+1}-1}: I_{J_{n+1}, k} \subset I_{J_n, K}} (\omega_{J_{n+1}, k}(Z))^{1/H_{n+1}} \prod_{m=0}^n (\omega_{J_m, K_m(k)}(Z))^{1/H_m-1/H_{m+1}} \\ = (\omega_{J_n, K}(Z))^{1/H_n} \prod_{m=0}^{n-1} (\omega_{J_m, K_m(k)}(Z))^{1/H_m-1/H_{m+1}},$$

Then  $f_{n+1}$  is a refinement on  $f_n$ : For every  $k \in \{K2^{J_{n+1}-J_n}, \dots, (K+1)2^{J_{n+1}-J_n} - 1\}$  and for  $t \in I_{J_{n+1}, k}$

$$f_{n+1}(t) = f_n(K2^{-J_n}) \\ + \sum_{k'=K2^{J_{n+1}-J_n}}^{k-1} (\omega_{J_{n+1}, k'}(Z))^{1/H_{n+1}} \prod_{m=0}^n (\omega_{J_m, K_m(k')}(Z))^{1/H_m-1/H_{m+1}} \\ + (\omega_{J_{n+1}, k}(Z))^{1/H_{n+1}} \prod_{m=0}^n (\omega_{J_m, K_m(k)}(Z))^{1/H_m-1/H_{m+1}} (2^{J_{n+1}} t - k).$$

Remark that for every  $(k, k') \in \{K2^{J_{n+1}-J_n}, \dots, (K+1)2^{J_{n+1}-J_n} - 1\}^2$ , for every  $m \in \{0, \dots, n\}$ ,  $K_m(k) = K_m(k')$ .

This can be achieved simultaneously on every dyadic interval  $I_{J_n, K}$ ,  $K \in \{0, \dots, 2^{J_n} - 1\}$ , by using the same generation  $J_{n+1}$  for the subdivision. The obtained function is again an increasing continuous function which is affine on every dyadic interval of generation  $J_{n+1}$ .

We then define  $g_{n+1}$  by  $g_{n+1}(y) = Z((f_{n+1})^{-1}(y))$  for  $y \in [0, 1]$ . Let  $U_{J_{n+1}, k}$  be the image of the interval  $I_{J_{n+1}, k}$  by  $f_{n+1}$ , for every  $k \in \{0, \dots, 2^{J_{n+1}} - 1\}$ . This function  $g_{n+1}$  maps any interval  $U_{J_{n+1}, k}$  to the interval  $Z(I_{J_{n+1}, k})$ , and thus satisfies:

$$\omega_{U_{J_{n+1}, k}}(g_{n+1}) = \omega_{J_{n+1}, k}(Z)$$

with

$$|U_{J_{n+1}, k}| = (\omega_{J_{n+1}, k}(Z))^{1/H_{n+1}} \prod_{m=0}^n (\omega_{J_m, K_m(k)}(Z))^{1/H_m-1/H_{m+1}}.$$

At this point, all the items of the iteration are ensured, except the item (4). We prove it now. As above, let us choose  $k \in \{0, \dots, 2^{J_{n+1}} - 1\}$  and  $K \in \{0, \dots, 2^{J_2} - 1\}$  are such that

$I_{J_{n+1},k} \subset I_{J_n,K}$ , and let  $k' \in \{0, \dots, 2^{J_{n+1}-J_n} - 1\}$  be such that  $k = K \cdot 2^{J_{n+1}-J_n} + k'$ . We have by (24)

$$\begin{aligned} |U_{J_{n+1},k}| &= (\omega_{J_n,K}(Z))^{1/H_0} (\omega_{J_{n+1}-J_n,k'}(Z_{J_n,K}))^{1/H_{n+1}} \\ (26) \qquad &= |U_{J_n,k}| (\omega_{J_{n+1}-J_n,k'}(Z_{J_n,K}))^{1/H_{n+1}}. \end{aligned}$$

Then, by (19),

$$|U_{J_{n+1},k}| \geq |U_{J_n,k}| 2^{-(J_{n+1}-J_n)\beta/H_n} = |U_{J_n,k}| 2^{-[J_n \eta_{J_n}] \beta / H_{n+1}}.$$

As above, since by (17) that  $|U_{J_n,k}| \leq 2^{-J_n \alpha / H_n}$ , we have

$$|U_{J_n,k}| \geq |U_{J_{n+1},k}| \geq |U_{J_n,k}|^{1 + \eta_{J_n} \frac{\beta H_n}{\alpha H_{n+1}}}.$$

**3.4. Convergence of  $(g_n)_{n \geq 0}$  and  $(f_n)_{n \geq 0}$ .** The convergence of the sequence  $(f_n)$  to a function  $f$  is almost immediate. Indeed, each  $f_n$  is an increasing function from  $[0, 1]$  to  $[0, 1]$ , and by item (5) of the iteration procedure, for every  $j \geq 1$ , for every  $k \in \{0, \dots, 2^j - 1\}$ ,  $f_m(k2^{-j})$  is constant as soon as  $J_m \geq j$ .

Recall that for every  $m$  and  $k$ ,  $|f_m(I_{J_m,k})| = |U_{J_m,k}|$ . By (26), and using (19), we obtain that  $|U_{J_{m+1},k}| \leq |U_{J_m,k}| 2^{-(J_{m+1}-J_m)\alpha}$ , and iteratively

$$(27) \qquad \text{for every } m \geq 1, |U_{J_m,k}| \leq C 2^{-J_m \alpha},$$

for some constant  $C$ . Hence the sequence  $(|U_{J_{m+1},k}|)_{m \geq 1}$  converge exponentially fast to zero, with an upper bound independent of  $k$ .

As a consequence, if  $m \geq n$ , then

$$\begin{aligned} \|f_n - f_m\|_\infty &\leq \max_{k \in \{0, \dots, 2^{J_m-1}\}} |f_m(I_{J_m,k})| \leq \max_{k \in \{0, \dots, 2^{J_m-1}\}} |U_{J_m,k}| \\ &\leq C 2^{-J_m \alpha}. \end{aligned}$$

This Cauchy criterion immediately gives the uniform convergence of the function series  $(f_n)$  to a continuous function  $f$ , whose value at each dyadic number is known as explained just above. The limit function  $f$  is also strictly increasing, since it is strictly increasing on the dyadic numbers.

The convergence of the functions sequence  $(g_n)_{n \geq 0}$  is then straightforward. Indeed, each  $f_n$  is an homeomorphism of  $[0, 1]$ , and admits a continuous inverse  $f_n^{-1}$ . We thus have, for every  $n \geq 1$ ,  $g_n = Z \circ f_n^{-1}$ . The series  $(f_n^{-1})$  also converges uniformly on  $[0, 1]$ . Since  $Z$  is uniformly continuous on  $[0, 1]$ ,  $(g_n)$  converges uniformly to a continuous function  $g : [0, 1] \rightarrow [0, 1]$ .

Remark that  $f$  also admits an inverse function  $f^{-1}$ , and that  $g = Z \circ f^{-1}$ .

**3.5. Properties of  $g$  and  $f$ .** Obviously,  $f$  is a strictly increasing function from  $[0, 1]$  to  $[0, 1]$ , which is what we were looking for. All we have to prove the monofractality property of  $g$ . This will follow from Lemma 2.2.

It has been noticed before that if we set, for every  $n \geq 0$ ,  $T_n = \{U_{J_n,k} : k \in \{0, \dots, 2^{J_n} - 1\}\}$ , then every  $T_n$  forms a covering of  $[0, 1]$  constituted by pairwise distinct intervals. We obviously have:

- $\lim_{n \rightarrow +\infty} \max_{T \in T_n} |T| = 0$  (using the remarks of Section 3.4 above),
- $(T_n)$  is a nested sequence of intervals,
- by item (4) of the iteration procedure, if  $T \in T_n \subset T' \in T_{n-1}$ , then we have  $|T'|^{1+Z_n} \leq |T| \leq |T'|$ , with  $Z_n = \eta_{J_{n-1}} \frac{\beta H_{n-1}}{\alpha H_n}$ . This sequence  $(Z_n)$  converges to zero, since  $(\eta_n)$  converges to zero and  $(H_n)$  converges to  $H$ .

In order to apply Lemma 2.2 and to get the monofractality property of  $g$ , it is thus enough to prove the last required properties, i.e. there is a positive sequence  $(\kappa_n)$  converging to zero such that for every  $T \in T_n$ ,  $|T|^{H+\kappa_n} \leq \omega_T(g) \leq |T|^{H-\kappa_n}$ .

For this, let  $n \geq 1$  and  $T \in T_n$ . This interval  $T$  can be written  $U_{J_n,k}$  for some  $k \in \{0, \dots, 2^{J_n}-1\}$ . We have  $|U_{J_n,k}| = (\omega_{J_n,k}(Z))^{1/H_n} \prod_{m=0}^{n-1} (\omega_{J_m, K_m(k)}(Z))^{1/H_m - 1/H_{m+1}}$  by construction, and  $g(U_{J_n,k}) = g_n(U_{J_n,k}) = \omega_{J_n,k}(Z)$ . We just have to verify that  $|U_{J_n,k}|^{H+\kappa_n} \leq \omega_{J_n,k}(Z) \leq |U_{J_n,k}|^{H-\kappa_n}$ , for some  $\kappa_n > 0$  independent of  $k$ .

We have

$$\log |U_{J_n,k}| = \frac{1}{H_n} \log \omega_{J_n,k}(Z) + \sum_{m=0}^{n-1} \left( \frac{1}{H_m} - \frac{1}{H_{m+1}} \right) \log \omega_{J_m, K_m(k)}(Z)$$

Writing that  $\left| \frac{1}{H_m} - \frac{1}{H_{m+1}} \right| \leq \frac{1}{H^2} (\varepsilon_{J_m} + \varepsilon_{J_{m+1}} + o(\varepsilon_{J_m})) \leq \frac{2}{H^2} (\varepsilon_{J_m} + o(\varepsilon_{J_m}))$  and  $\frac{1}{H_n} \leq \frac{1}{H} (1 + \frac{\varepsilon_{J_n}}{H} + o(\varepsilon_{J_n}))$ , we obtain

$$(28) \quad \left| \frac{\log |U_{J_n,k}|}{\log \omega_{J_n,k}(Z)} - \frac{1}{H} \right| \leq \frac{\varepsilon_{J_n}}{H^2} + \frac{2}{H^2} \frac{\sum_{m=0}^{n-1} (\varepsilon_{J_m} + o(\varepsilon_{J_m})) \log \omega_{J_m, K_m(k)}(Z)}{\log \omega_{J_n,k}(Z)}$$

Let us denote  $d_{m,k} = -\log \omega_{J_m, K_m(k)}(Z)$ , and  $\psi_m = \varepsilon_{J_m}$  for every  $m$  and  $k$ . Comparing the last inequality with the desired result, all we have to show is that

$$(29) \quad \frac{\sum_{m=0}^{n-1} \psi_m d_{m,k}}{d_{n,k}} \rightarrow 0 \quad \text{when } n \rightarrow +\infty,$$

independently of  $k$ .

This is obtained as follows: Start from  $J_1$ , that we suppose (without loss of generality) to be greater than 100. Recall that, by the remarks made at the beginning of Section 3, we assumed that for every  $n \geq 1$ ,  $\eta_{J_n} \geq (\log J_n)^{-1}$ . Subsequently, every term  $J_n$  is greater than  $l_n$ , where  $(l_n)$  is the sequence defined recursively by  $l_{n+1} = l_n(1 + 1/\log l_n)$  and  $l_1 = 100$ . Let us study the growth rate of such a sequence. It is obvious that  $\lim_{n \rightarrow +\infty} l_n = +\infty$ . We set  $v_n = \log l_n$ . We have  $v_{n+1} = v_n + \log(1 + 1/v_n) \geq v_n + (1 - \varepsilon)/v_n$  for every  $n$ , with  $\varepsilon$  that can be taken less than  $1/4$  since  $v_1$  is large enough. In particular, since  $v_1 \geq \sqrt{2}$ ,  $v_2 \geq \sqrt{2} + (1 - \varepsilon)/\sqrt{2} \geq \sqrt{3}$ . Recursively, if we assume that  $v_n \geq \sqrt{n+1}$ , then  $v_{n+1} \geq \sqrt{n} + (1 - \varepsilon)/\sqrt{n} \geq \sqrt{n+1}$ . Hence the sequence  $(l_n)$  converges to  $+\infty$  faster than  $\exp \sqrt{n}$ , and thus faster than any polynomial  $n^\delta$ . (We could be more precise, and prove using the same arguments that the growth rate of  $(l_n)$  is exactly  $\exp \sqrt{n}$ .)

Let us now find an upper bound for  $\psi_m$ . Using the item (1) of condition **C2**, we have that  $\varepsilon_j = o\left(\frac{1}{(\log j)^{2+\kappa}}\right)$ . Using the lower bound we found for  $J_m$  with  $\varepsilon$  chosen small enough, we get that  $\psi_m = o(n^{-1/2(2+\kappa)}) \leq o(n^{-1-\kappa/2})$ .

The crucial point is that  $\sum_{m \geq 1} \psi_m < +\infty$ . Now, since  $d_{m,k}$  is a sequence increasing toward  $+\infty$ , rewrite the left term of (28) as  $\sum_{m=0}^{n-1} \psi_m \frac{d_{m,k}}{d_{n,k}}$ , where  $0 \leq \frac{d_{m,k}}{d_{n,k}} \leq 1$ . By a classical Caesaro method, we get that (29) is true, independtly of  $k$ .

This directly implies, by (28), that independtly of  $k$ ,  $\left| \frac{\log |U_{J_n,k}|}{\log \omega_{J_n,k}(Z)} - \frac{1}{H} \right| \leq \kappa_n$ , for some sequence  $\kappa_n$  that converges to zero.

We now apply Lemma 2.2, which implies that  $g$  is monofractal with exponent  $H$ .

## 4. THE CASE OF CLASSICAL MONOFRactal FUNCTIONS AND PROCESSES

It is satisfactory to check that classical monofractal functions verify the conditions of Theorem 1.7, and that the exponent  $H(Z)$  is actually equal to their monofractal exponent. The proofs below are also representative examples of the method used to get convergence rates for  $H_j(Z)$  to  $H(Z)$ .

**4.1. Weierstrass-type functions.** Let  $0 < \alpha < 1$ ,  $\beta > \alpha$  and  $b > 1$  be three real numbers. Let  $w$  be a bounded function that belongs to the global Hölder class  $C^\beta((0, 1))$ . Consider the Weierstrass-type function

$$(30) \quad Z(t) = \sum_{k=0}^{\infty} b^{-\alpha k} w(b^k t).$$

By [5], either the function  $Z$  is  $C^\beta$ , or it is monofractal with exponent  $\alpha$ . For  $w(t) = \sin(t)$ , we obtain the classical Weierstrass functions monofractal with exponent  $\alpha$ . In fact, it is proved in [5] that, if  $Z \notin C^\beta$  (which is our assumption from now on), then there is a constant  $C > 1$  such that

$$(31) \quad C^{-1} 2^{-j\alpha} \leq \omega_{j,k}(Z) \leq C 2^{-j\alpha}.$$

As a direct consequence,  $C^{-1/\alpha} \leq \sum_{k=0}^{2^j-1} (\omega_{j,k}(Z))^{1/\alpha} \leq C^{1/\alpha}$ , and obviously  $H(Z) = \alpha$ .

Let us find the convergence rate of  $H_j(Z)$  toward  $H(Z)$ . We are looking for a value of  $\varepsilon > 0$  and for a scale  $J_0$  for which  $\sum_{k=0}^{2^j-1} (\omega_{j,k}(Z))^{1/(\alpha+\varepsilon)} > 1$ , for every  $j \geq J_0$ . Let  $j \geq 1$ . We have, by (31),

$$\sum_{k=0}^{2^j-1} (\omega_{j,k}(Z))^{1/(\alpha+\varepsilon)} \geq 2^j C^{-1/(\alpha+\varepsilon)} 2^{-j\alpha/(\alpha+\varepsilon)}.$$

For  $\varepsilon$  small,  $1/(\alpha + \varepsilon) = 1/\alpha - \varepsilon/\alpha^2 + o(\varepsilon)$ , and thus our constraint is reached as soon as  $1 < C^{-1/\alpha + \varepsilon/\alpha^2 + o(\varepsilon)} 2^{j\varepsilon/\alpha + o(j\varepsilon)}$ . This leads to

$$j\varepsilon/\alpha + o(j\varepsilon) + (\log_2 C)(-1/\alpha + \varepsilon/\alpha^2 + o(\varepsilon)) > 0.$$

There is a generation  $J_0$  such that the last inequality is realized by  $\varepsilon = \frac{2 \log_2 C}{J_0}$ . Subsequently, one necessarily has  $1/H_j(Z) \geq 1/(\alpha + \varepsilon)$  for every  $j \geq J_0$ , since

$$\sum_{k=0}^{2^j-1} (\omega_{j,k}(Z))^{1/(\alpha+\varepsilon)} > \sum_{k=0}^{2^j-1} (\omega_{j,k}(Z))^{1/H_j(Z)} = 1$$

and the mapping  $h \rightarrow \sum_{k=0}^{2^j-1} (\omega_{j,k}(Z))^{1/h}$  is increasing with  $h$ . Hence  $H_j(Z) \leq \alpha + \varepsilon$ .

Using the same method, we obtain  $H_j(Z) \geq \alpha - 2\varepsilon$  for  $j \geq J_0$ .

Finally, we have found  $J_0$  large enough so that for every  $j \geq J_0$ ,  $|H_j(Z) - \alpha| \leq \varepsilon_0$ , where we have set  $\varepsilon_0 = 2\varepsilon = 4 \log_2 C / (J_0)$ .

For every  $J \geq 1$  and  $K \in \{0, \dots, 2^J - 1\}$ , we easily get the same convergence rates of  $H_j(Z_{J,K})$  toward  $\alpha$  from the self-affinity property of the Weierstrass functions. More precisely, fix  $J$  and  $K$ , and let  $j \geq J + 1$ . Remark that by construction of  $Z_{J,K}$ , we have  $\omega_{j-J,k}(Z_{J,K}) = \frac{\omega_{j,K 2^{-J} + k}(Z)}{\omega_{J,K}(Z)}$ . We are looking for a value of  $\varepsilon$  for which

$$\sum_{k=0, \dots, 2^j-1: I_{j,k} \subset I_{J,K}} \left( \frac{\omega_{j,k}(Z)}{\omega_{J,K}(Z)} \right)^{1/(\alpha+\varepsilon)} > 1,$$



for every  $j$  large enough. By (31) (used two times), and remarking that there are  $2^{j-J}$  dyadic intervals of generation  $j$  included in  $I_{J,K}$ , we get

$$\sum_{k=0, \dots, 2^j-1: I_{j,k} \subset I_{J,K}} \left( \frac{\omega_{j,k}(Z)}{\omega_{J,K}(Z)} \right)^{1/(\alpha+\varepsilon)} \geq 2^{j-J} C^{-2/(\alpha+\varepsilon)} 2^{-(j-J)\alpha/(\alpha+\varepsilon)}.$$

The same computations as above yield that, if we impose  $\eta_J = 1/\log_2 J$  and  $\varepsilon_J = 4(\log_2 C)(\log_2 J)/J$ , then for every  $j \geq J + J\eta_J$ ,  $H_{j-J}(Z_{J,K}) \leq \alpha - \varepsilon_J$ . Similarly, we obtain  $H_j(Z) \geq \alpha + 2\varepsilon_J$  for  $j \geq J + J\eta_J$ .

Finally, for every  $j \geq J + J\eta_J$ ,  $|H_{j-J}(Z_{J,K}) - \alpha| \leq \varepsilon_J$ , and  $\varepsilon_j = o(1/(\log j)^{2+\kappa})$ .

Consequently, the Weierstrass functions satisfy **C1** and **C2** with  $H = \alpha$ , and they are also monofractal from our viewpoint.

**4.2. Sample paths of Brownian motions and fractional Brownian motions.** Classical estimations on the oscillations of sample paths of Brownian motions  $(B_t)_{t \geq 0}$  yield ([11])

$$\begin{aligned} \mathbb{P} \left( \omega_{j,k}(B_t) \leq j2^{-j/2} \right) &\leq \frac{1}{2\pi} \exp(-j^2\pi^2) \\ \mathbb{P} \left( \omega_{j,k}(B_t) \geq \frac{1}{j}2^{-j/2} \right) &\leq \frac{4j}{2\pi} \exp(-j^2/8) \end{aligned}$$

Hence, by a classical Borel-Cantelli argument, with probability one, there is a generation  $J_c$  such that for every  $j \geq J_c$ , we have the bounds  $\frac{1}{j}2^{-j/2} \leq \omega_{j,k}(B_t) \leq j2^{-j/2}$  for the oscillations.

The same computations as for the Weierstrass functions show that there is a generation  $J_0$  such that if  $j \geq J_0 \geq J_c$ , then

$$\sum_{k=0}^{2^j-1} (\omega_{j,k}(B_t))^{1/(1/2+\varepsilon_{J_0})} > 1 \text{ and } \sum_{k=0}^{2^j-1} (\omega_{j,k}(B_t))^{1/(1/2-\varepsilon_{J_0})} < 1,$$

where  $\varepsilon_{J_0} \geq C \frac{\log J_0}{J_0}$  (for some suitable constant  $C$ ). As a consequence,  $|H_j(B_t) - 1/2| \leq \varepsilon_{J_0}$  for every  $j \geq J_0$ .

The self-similarity property of Brownian motions yields that for every  $J \geq 1$  and  $K \in \{0, \dots, 2^J - 1\}$ , for every  $j \geq J/\log J$ ,  $|H_j((B_t)_{J,K}) - 1/2| \leq \varepsilon_J$ , where  $\varepsilon_J = C \frac{\log^2 J}{J}$ , for some constant  $C$  independent of  $J$  and  $K$ . We omit the details here, that can be easily checked by the reader.

Consequently, a sample path of Brownian motion satisfies with probability one **C1** and **C2**, with  $H(B_t) = 1/2$ .

Similar estimations on the oscillations of fractional Brownian motions  $B_h$  of Hurst exponent  $h$  lead to the same almost sure result for the sample paths, which also satisfy almost surely **C1** and **C2** with  $H(B_h) = h$ .

## 5. APPLICATIONS TO SELF-SIMILAR FUNCTIONS: THEOREM 1.10

We consider the class of self-similar functions defined in Definition 1.9, with the parameters of  $Z$  and the contractions  $S_k$  satisfying (11).

The multifractal analysis of such a function  $Z$  is performed in [9]. Here we are going to prove that, under the conditions (11) on the  $\lambda_k$  and the  $S_k$ ,  $Z$  is CMT, and that the multifractal behavior of  $Z$  can be directly deduced from this analysis. It is a case where our analysis provides a natural way to compute the singularity spectrum of  $Z$ .

**5.1. Preliminary results on the oscillations of  $Z$ .** Let us introduce some notations: for every  $n \geq 1$ , for every  $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) \in \{0, 1, \dots, d-1\}^n$ , we denote  $I_{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n}$  the interval  $S_{\varepsilon_1} \circ S_{\varepsilon_2} \circ \dots \circ S_{\varepsilon_n}([0, 1])$ . The integer  $n$  being given, the open intervals  $(I_{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n})$  are pairwise disjoint, and the union of the closed intervals  $I_{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n}$  equals  $[0, 1]$ . Now fix an integer  $n \geq 1$  and a sequence  $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) \in \{0, 1, \dots, d-1\}^n$ . The interval  $I_{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n}$  has a length equal to  $r_{\varepsilon_1} r_{\varepsilon_2} \cdots r_{\varepsilon_n}$ . Finally, by iterating  $n$  times formula (12), we get that for every  $t \in I_{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n}$ ,

$$(32) \quad \begin{aligned} Z(t) &= \lambda_{\varepsilon_1} \cdot \lambda_{\varepsilon_2} \cdots \lambda_{\varepsilon_n} \cdot (Z \circ S_{\varepsilon_n}^{-1} \circ S_{\varepsilon_{n-1}}^{-1} \circ \dots \circ S_{\varepsilon_1}^{-1})(t) \\ &+ \lambda_{\varepsilon_1} \cdot \lambda_{\varepsilon_2} \cdots \lambda_{\varepsilon_{n-1}} \cdot (\phi \circ S_{\varepsilon_{n-1}}^{-1} \circ S_{\varepsilon_{n-2}}^{-1} \circ \dots \circ S_{\varepsilon_1}^{-1})(t) \\ &+ \dots \\ &+ \lambda_{\varepsilon_1} \cdot \lambda_{\varepsilon_2} \cdot (\phi \circ S_{\varepsilon_2}^{-1} \circ S_{\varepsilon_1}^{-1})(t) \\ &+ \lambda_{\varepsilon_1} \cdot (\phi \circ S_{\varepsilon_1}^{-1})(t) \\ &+ \phi(t). \end{aligned}$$

Recall that  $\chi_{\max}$  is defined in (11).

**Proposition 5.1.** *Let  $\kappa = \frac{\chi_{\max}}{1 - \chi_{\max}}$ . Then either  $Z$  is a  $\kappa$ -Lipschitz function, or there is a constant  $C > 1$  such that for every  $n \geq 1$ , for every  $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) \in \{0, 1, \dots, d-1\}^n$ ,*

$$(33) \quad C^{-1} \cdot |\lambda_{\varepsilon_1} \cdot \lambda_{\varepsilon_2} \cdots \lambda_{\varepsilon_n}| \leq \omega_{I_{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n}}(Z) \leq C \cdot |\lambda_{\varepsilon_1} \cdot \lambda_{\varepsilon_2} \cdots \lambda_{\varepsilon_n}|.$$

*Proof.* We first find an upper-bound for  $\omega_{I_{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n}}(Z)$ . We use the iterated formula (32).

Let  $n$  and  $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) \in \{0, 1, \dots, d-1\}^n$ . Remark that when  $t$  ranges in  $I_{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n}$ ,  $(S_{\varepsilon_n}^{-1} \circ S_{\varepsilon_{n-1}}^{-1} \circ \dots \circ S_{\varepsilon_1}^{-1})(t)$  ranges in  $[0, 1]$ . Hence the oscillation of the first term of (32) is upper-bounded by  $|\lambda_{\varepsilon_1} \lambda_{\varepsilon_2} \cdots \lambda_{\varepsilon_n}| \cdot \omega_{[0,1]}(Z)$ .

Now, for every  $k \in \{1, \dots, n-1\}$ , when  $t$  ranges in  $I_{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n}$ ,  $(S_{\varepsilon_k}^{-1} \circ S_{\varepsilon_{k-1}}^{-1} \circ \dots \circ S_{\varepsilon_1}^{-1})(t)$  ranges in  $I_{\varepsilon_{k+1}, \dots, \varepsilon_n}$ . Using that  $\phi$  is a Lipschitz function, we get that the oscillation of each term of the form  $\lambda_{\varepsilon_1} \lambda_{\varepsilon_2} \cdots \lambda_{\varepsilon_k} \cdot (\phi \circ S_{\varepsilon_k}^{-1} \circ S_{\varepsilon_{k-1}}^{-1} \circ \dots \circ S_{\varepsilon_1}^{-1})(t)$  is upper bounded by  $|\lambda_{\varepsilon_1} \lambda_{\varepsilon_2} \cdots \lambda_{\varepsilon_k}| (r_{\varepsilon_{k+1}} \cdots r_{\varepsilon_n})$ . Finally, we obtain using (11)

$$\begin{aligned} \omega_{I_{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n}}(Z) &\leq |\lambda_{\varepsilon_1} \lambda_{\varepsilon_2} \cdots \lambda_{\varepsilon_n}| + \sum_{k=1}^{n-1} |\lambda_{\varepsilon_1} \lambda_{\varepsilon_2} \cdots \lambda_{\varepsilon_k}| \cdot (r_{\varepsilon_{k+1}} \cdots r_{\varepsilon_n}) \\ &\leq |\lambda_{\varepsilon_1} \lambda_{\varepsilon_2} \cdots \lambda_{\varepsilon_n}| \cdot \left[ 1 + \sum_{k=1}^{n-1} \left( \prod_{j=1}^k \frac{r_j}{|\lambda_j|} \right) \right] \\ &\leq |\lambda_{\varepsilon_1} \lambda_{\varepsilon_2} \cdots \lambda_{\varepsilon_n}| \cdot \left[ 1 + \sum_{k=1}^{n-1} \chi_{\max}^k \right] \leq C_1 \cdot |\lambda_{\varepsilon_1} \lambda_{\varepsilon_2} \cdots \lambda_{\varepsilon_n}|, \end{aligned}$$

where  $C_1 = 1 + \sum_{k=1}^{+\infty} \chi_{\max}^k < +\infty$ .

We now move to the lower bound. Assume that  $Z$  is not  $\kappa$ -Lipschitz. There are two real numbers  $0 \leq t_0, t'_0 \leq 1$  such that  $|Z(t'_0) - Z(t_0)| \geq (\kappa + \eta)|t'_0 - t_0|$ , for some  $\eta > 0$ . Let  $n$  and  $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) \in \{0, 1, \dots, d-1\}^n$ . Let us call  $t_n = S_1 \circ S_2 \circ \dots \circ S_n(t_0)$  and  $t'_n = S_1 \circ S_2 \circ \dots \circ S_n(t'_0)$ . We obviously have  $t_n, t'_n \in I_{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n}$ , and thus  $\omega_{I_{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n}}(Z) \geq |Z(t'_n) - Z(t_n)|$ . Using again (32), we get by the same lines of

computations as above

$$\begin{aligned}
& |Z(t'_n) - Z(t_n)| \\
& \geq |\lambda_{\varepsilon_1} \lambda_{\varepsilon_2} \cdots \lambda_{\varepsilon_n}| \cdot |Z(t'_0) - Z(t_0)| - \sum_{k=1}^{n-1} |\lambda_{\varepsilon_1} \lambda_{\varepsilon_2} \cdots \lambda_{\varepsilon_k}| (r_{\varepsilon_{k+1}} \cdots r_{\varepsilon_n}) |t'_0 - t_0| \\
& \geq |\lambda_{\varepsilon_1} \lambda_{\varepsilon_2} \cdots \lambda_{\varepsilon_n}| \cdot |Z(t'_0) - Z(t_0)| \cdot \left[ 1 - \sum_{k=1}^{n-1} \left( \prod_{j=1}^k \frac{r_j}{|\lambda_j|} \right) \frac{|t'_0 - t_0|}{|Z(t'_0) - Z(t_0)|} \right] \\
& \geq |\lambda_{\varepsilon_1} \lambda_{\varepsilon_2} \cdots \lambda_{\varepsilon_n}| \cdot |Z(t'_0) - Z(t_0)| \cdot \left[ 1 - \frac{1}{\kappa + \eta} \sum_{k=1}^{+\infty} \chi_{\max}^k \right] \geq C_2 \cdot |\lambda_{\varepsilon_1} \lambda_{\varepsilon_2} \cdots \lambda_{\varepsilon_n}|,
\end{aligned}$$

where  $C_2 = |Z(t'_0) - Z(t_0)| \left(1 - \frac{1}{\kappa + \eta} \frac{\chi_{\max}}{1 - \chi_{\max}}\right) > 0$  by assumption.

Finally, (33) is proved with  $C = \max(C_1, C_2^{-1})$ .  $\square$

**5.2. Comparison of  $Z$  with a self-similar measure.** In order to prove that the function  $Z$  (12) satisfies our conditions **C1** and **C2**, we introduce a self-similar measure  $\mu$ , whose multifractal behavior will be compared with the one of  $Z$ , and the notion of multifractal formalism.

Let us consider the exponent  $\beta > 1$  such that (13) holds and the associated self-similar measure  $\mu$  defined by (14)  $\mu = \sum_{k=0}^{d-1} p_k \cdot (\mu \circ S_k^{-1})$ . In our case where the similitudes do not overlap, it is easily checked that by construction, for every  $n$  and  $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) \in \{0, 1, \dots, d-1\}^n$ , we have

$$(34) \quad \mu(I_{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n}) = p_{\varepsilon_1} p_{\varepsilon_2} \cdots p_{\varepsilon_n} = |\lambda_{\varepsilon_1} \lambda_{\varepsilon_2} \cdots \lambda_{\varepsilon_n}|^\beta.$$

This class of measures has been extensively studied [6, 7, 16, 18]. For instance, the multifractal analysis of  $\mu$  is very well known. For this, let us introduce the so-called  $L^q$ -spectrum of  $\mu$  defined by

$$(35) \quad \tau_\mu : q \in \mathbb{R} \mapsto \tau_\mu(q) = \liminf_{j \rightarrow +\infty} \tau_\mu(j, q), \quad \text{where } \tau_\mu(j, q) = \frac{\log_2 \sum_{k=0}^{2^j-1} \mu(I_{j,k})^q}{-j}.$$

We only recall the properties we need [7, 16, 18]

**Proposition 5.2.** (1) For every  $q \in \mathbb{R}$ ,  $\tau_\mu(q)$  is the unique real number satisfying the equation  $\sum_{k=0}^{d-1} (p_k)^q (r_k)^{\tau_\mu(q)} = 1$ . The mapping  $q \mapsto \tau_\mu(q)$  is analytic on its support. Moreover, the liminf used to define  $\tau_\mu(q)$  is in fact a limit for every  $q$  such that  $\tau_\mu(q)$  is finite.

(2) There is an interval of exponents  $I_\mu = [\alpha_{\min}, \alpha_{\max}]$  such that for every  $\alpha \in I_\mu$ ,  $d_\mu(\alpha) = (\tau_\mu)^*(\alpha)$ , where  $(\tau_\mu)^*(\alpha) := \inf_{q \in \mathbb{R}} (q\alpha - \tau_\mu(q))$  is by definition the Legendre transform of  $\tau_\mu$ .

(3) If  $\alpha \notin I_\mu$ , then  $\{x : \alpha_\mu(t) = \alpha\} = \emptyset$ .

(4) There is  $M \geq 1$  such that for every  $n$ , for every  $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) \in \{0, 1, \dots, d-1\}^n$ ,  $|I_{\varepsilon_1, \dots, \varepsilon_n}|^M \leq \mu(I_{\varepsilon_1, \dots, \varepsilon_n}) \leq |I_{\varepsilon_1, \dots, \varepsilon_n}|^{1/M}$ .

Part (2) above is known as the multifractal formalism for measures, when it holds.

Let us come back to the function  $Z$ . We are going to show that  $Z$  satisfies **C1** and **C2** in the case where all the similitudes  $S_i$  send  $[0, 1]$  to dyadic intervals. For more general similitudes, some efforts are needed to conclude, and are left to the reader. Nevertheless, next Section 5.3 proposes a different proof of Theorem 1.10 (for any self-similar function satisfying our conditions, not only the dyadic ones).

Each dyadic interval  $I_{j,k}$  is included in one dyadic interval  $I_{\varepsilon_1, \dots, \varepsilon_n}$ , and contains a dyadic interval  $I_{\varepsilon_1, \dots, \varepsilon_n, \varepsilon_{n+1}}$ , such that  $I_{\varepsilon_1, \dots, \varepsilon_n}$  and  $I_{\varepsilon_1, \dots, \varepsilon_n, \varepsilon_{n+1}}$  can be written respectively  $I_{j',k'}$  and  $I_{j'',k''}$  with  $0 \leq j - j', j'' - j' \leq C$ , for some constant  $C$  independent of  $j$  and  $k$ . Consequently,  $\omega_{I_{\varepsilon_1, \dots, \varepsilon_n, \varepsilon_{n+1}}}(Z) \leq \omega_{I_{j,k}}(Z) \leq \omega_{I_{\varepsilon_1, \dots, \varepsilon_n}}(Z)$ , and thus

$$C^{-1} \mu(I_{\varepsilon_1, \dots, \varepsilon_n, \varepsilon_{n+1}})^{1/\beta} \leq \omega_{j,k}(Z) \leq C \mu(I_{\varepsilon_1, \dots, \varepsilon_n})^{1/\beta}.$$

Using now the self-similarity properties of the measure and the open set condition, we see that  $\max_k(p_k) \cdot \mu(I_{\varepsilon_1, \dots, \varepsilon_n, \varepsilon_{n+1}}) \geq \mu(I_{j,k})$  and  $\mu(I_{j,k}) \geq \min_k(p_k) \cdot \mu(I_{\varepsilon_1, \dots, \varepsilon_n})$ . Hence, combining this with the last double inequality, we obtain that for every  $j$  and  $k$

$$(36) \quad C^{-1} \mu(I_{j,k})^{1/\beta} \leq \omega_{j,k}(Z) \leq C \mu(I_{j,k})^{1/\beta}$$

for another constant  $C$ , i.e. Proposition 5.1 extends to all dyadic intervals.

Let us check that  $Z$  satisfies conditions **C1** and **C2**. Remark that, because of (36),

$$C^{-\beta} = C^{-\beta} \sum_{k=0}^{2^j-1} \mu(I_{j,k}) \leq \sum_{k=0}^{2^j-1} (\omega_{j,k}(Z))^\beta \leq C^\beta \sum_{k=0}^{2^j-1} \mu(I_{j,k}) = C^\beta.$$

Let  $\varepsilon_1 > 0$ . Using part (4) of Proposition 5.2 to find upper- and lower-bounds for  $\omega_{j,k}(Z)$  uniformly in  $k$ , we obtain

$$\begin{aligned} \sum_{k=0}^{2^j-1} (\omega_{j,k}(Z))^{\beta-\varepsilon_1} &\geq \sum_{k=0}^{2^j-1} (\omega_{j,k}(Z))^\beta (C \mu(I_{j,k}))^{-\varepsilon_1} \geq C^{-\beta-\varepsilon_1} 2^{j\varepsilon_1/M} \\ \text{and} \quad \sum_{k=0}^{2^j-1} (\omega_{j,k}(Z))^{\beta+\varepsilon_1} &\leq \sum_{k=0}^{2^j-1} (\omega_{j,k}(Z))^\beta (C \mu(I_{j,k}))^{\varepsilon_1} \geq C^{\beta+\varepsilon_1} 2^{-j\varepsilon_1 M}. \end{aligned}$$

Hence, the same computations as in the case of Weierstrass functions lead to the following choice: for some  $J_0$  large enough, we set  $\varepsilon_1 = \frac{2\beta M \log_2 C}{\log J_0}$ , and thus for every  $j \geq J_0$ ,  $|H_j(Z) - H(Z)| \leq \varepsilon_1$ .

Let now  $J, K$  be two integers,  $\varepsilon > 0$ , and focus on  $H(Z_{J,K})$ . The same computations as above and as in the Weierstrass case yield for  $j \geq J$

$$\sum_{k'=0}^{2^{j-J}-1} (\omega_{j-J,k'}(Z_{J,K}))^{\beta-\varepsilon} = \sum_{k=0: I_{j,k} \subset I_{J,K}}^{2^j-1} \left( \frac{\omega_{j,k}(Z)}{\omega_{J,K}(Z)} \right)^{\beta-\varepsilon}.$$

First notice that  $\left( \frac{1}{\omega_{J,K}(Z)} \right)^{\beta-\varepsilon} \geq \left( \frac{1}{C \mu(I_{J,K})^{1/\beta}} \right)^{-\beta+\varepsilon} \geq C^{-\beta-\varepsilon} \mu(I_{J,K})^{-1+\varepsilon/\beta}$ . Then we remark that

$$\sum_{k=0: I_{j,k} \subset I_{J,K}}^{2^j-1} (\omega_{j,k}(Z))^{\beta-\varepsilon} \geq \sum_{k=0: I_{j,k} \subset I_{J,K}}^{2^j-1} (\omega_{j,k}(Z))^\beta (C \mu(I_{j,k}))^{-\varepsilon}.$$

Combining these inequalities we get

$$\sum_{k'=0}^{2^{j-J}-1} (\omega_{j-J,k'}(Z_{J,K}))^{\beta-\varepsilon} \geq C^{-\beta-2\varepsilon} \sum_{k=0: I_{j,k} \subset I_{J,K}}^{2^j-1} \frac{(\omega_{j,k}(Z))^\beta \mu(I_{J,K})^{\varepsilon/\beta}}{\mu(I_{J,K}) \mu(I_{j,k})^\varepsilon}.$$

Let us focus on  $\frac{\mu(I_{J,K})^{\varepsilon/\beta}}{\mu(I_{j,k})^\varepsilon}$ . Since  $\beta > 1$ , we have  $\frac{\mu(I_{J,K})^{\varepsilon/\beta}}{\mu(I_{j,k})^\varepsilon} \geq \left( \frac{\mu(I_{J,K})}{\mu(I_{j,k})} \right)^\varepsilon$ . This quantity is lower bounded by  $L^{(j-J)\varepsilon}$  for some constant  $L$  (uniformly in  $k$  and  $K$ ), since the ratio

of the  $\mu$ -measures of a dyadic interval and its father (in the dyadic tree) is uniformly upper- and lower-bounded for our dyadic self-similar measure  $\mu$ . Finally, we obtain

$$\begin{aligned} \sum_{k'=0}^{2^j-1} (\omega_{j-J,k'}(Z_{J,K}))^{\beta-\varepsilon} &\geq C^{-\beta-2\varepsilon} \sum_{k=0: I_{j,k} \subset I_{J,K}}^{2^j-1} \frac{(\omega_{j,k}(Z))^{\beta}}{\mu(I_{J,K})} L^{(j-J)\varepsilon} \\ &\geq C^{-2\beta-2\varepsilon} \sum_{k=0: I_{j,k} \subset I_{J,K}}^{2^j-1} \frac{\mu(I_{j,k})}{\mu(I_{J,K})} L^{(j-J)\varepsilon} \\ &\geq C^{-2\beta-2\varepsilon} L^{(j-J)\varepsilon}. \end{aligned}$$

Hence if we fix  $\eta_J = 1/\log_2 J$ , then the sum above is greater than 1 as soon as  $j \geq J + [J\eta_J]$  and  $\varepsilon \geq \frac{4\beta \log C \log_2 J}{J \log L}$ . Thus for  $j \geq J + [J\eta_J]$ ,  $H_{j-J}(Z_{J,K}) - H(Z_{J,K}) \leq \frac{1}{\beta-\varepsilon} \leq \frac{1}{\beta} + \varepsilon_J$  with  $\varepsilon_J = \frac{8 \log C \log_2 J}{J\beta \log L}$ .

Similarly one shows that for  $j \geq J + [J\eta_J]$ ,  $H_{j-J}(Z_{J,K}) - H(Z_{J,K}) \geq \frac{1}{\beta} - \varepsilon_J$ , and **C2** holds true for  $Z$ .

Applying Theorem 1.7 yields that  $Z$  is CMT and can be written as  $Z = g \circ f$ , where  $g$  is monofractal of exponent  $1/\beta$ . In order to fully complete the proof of Theorem 1.10, we have to prove that if we are careful, the obtained function  $f$  can be taken equal to the integral of the measure  $\mu$ . There is a nice way to get this result, that does not necessitates to modify the construction step by step (which might be a very unpleasant work).

**5.3. A direct proof of the existence of the CMT decomposition in Theorem 1.10.** Here we propose a quick proof of the fact that a self-similar function  $Z$  satisfying the assumptions of Theorem 1.10 can be written  $Z = g \circ F$ , where  $F$  is the integral of  $\mu$  defined as  $F : t \in [0, 1] \mapsto \mu([0, t])$  and  $g$  is monofractal with exponent  $1/\beta$ .

Consider directly the mapping  $g$

$$g : t \in [0, 1] \mapsto Z \circ F^{-1}(t).$$

By construction, for every interval  $I_{\varepsilon_1, \dots, \varepsilon_n}$ ,  $|F(I_{\varepsilon_1, \dots, \varepsilon_n})| = \mu(I_{\varepsilon_1, \dots, \varepsilon_n})$ . Let us denote, for every  $n \geq 1$ , for every  $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) \in \{0, 1, \dots, d-1\}^n$ ,  $U_{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n} = F(I_{\varepsilon_1, \dots, \varepsilon_n})$ . When  $n$  is fixed, the set of intervals  $\mathcal{U}_n = \{U_{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n} : (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) \in \{0, 1, \dots, d-1\}^n\}$  forms a covering of  $[0, 1]$ .

The sequence of coverings  $(\mathcal{U}_n)_{n \geq 1}$  satisfies all the assumptions of Lemma 2.2. This is due to the fact that the intervals  $I_{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n}$  are embedded and to Part (4) of the above Proposition 5.2.

All we have to prove is that

$$(37) \quad |U_{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n}|^{1/\beta + \kappa_n} \leq \omega_{U_{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n}}(g) \leq |U_{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n}|^{1/\beta - \kappa_n},$$

for every interval  $U_{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n}$ . This is easily achieved as follows. We remark that

$$g(U_{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n}) = Z \circ F^{-1}(F(I_{\varepsilon_1, \dots, \varepsilon_n})) = Z(I_{\varepsilon_1, \dots, \varepsilon_n}).$$

Hence by Proposition 5.1, if  $Z$  is not  $\kappa$ -Lipschitz, for some constant  $C > 1$  we have

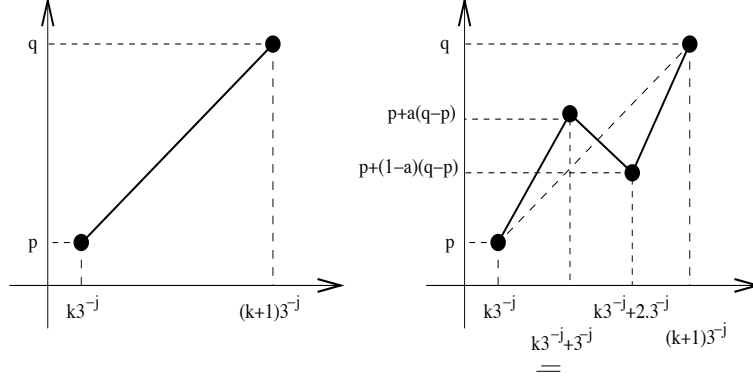
$$C^{-1} |\lambda_1 \cdots \lambda_n| \leq \omega_{U_{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n}}(g) = \omega_{I_{\varepsilon_1, \dots, \varepsilon_n}}(Z) \leq C |\lambda_1 \cdots \lambda_n|.$$

Combining this with (34), we find

$$C^{-1} \mu(I_{\varepsilon_1, \dots, \varepsilon_n})^{1/\beta} \leq \omega_{U_{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n}}(g) \leq C \mu(I_{\varepsilon_1, \dots, \varepsilon_n})^{1/\beta}.$$

As noticed above,  $|U_{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n}| = |F(I_{\varepsilon_1, \dots, \varepsilon_n})| = \mu(I_{\varepsilon_1, \dots, \varepsilon_n})$ , and thus (37) is proved.

Applying Lemma 2.2 yields the result.

FIGURE 1. Iterated construction of  $Z_a$ , from step  $j$  to step  $j + 1$ 

**5.4. Computation of the singularity spectrum of  $Z$ .** It is known that the self-similar measure  $\mu$  satisfies the multifractal formalism (we are in a case where the similitudes do not overlap). As a consequence of Section 7 above, for every  $h \geq 0$ , we have  $d_Z(h) = d_\mu(\beta h) = (\tau_\mu)^*(h)$ .

#### 6. AN EXAMPLE OF FUNCTION SATISFYING **C1-C2** IN A TRIADIC BASIS

We recall the construction of multifractal functions of [15], which somehow generalizes the Bourbaki's and Perkin's functions.

Let us consider the function  $Z_a$  defined for  $0 \leq a \leq 1$  as the limit of an iterated construction: Start from  $Z_a^0(t) = t$  on  $[0, 1]$ , and define  $Z_a^j(t)$  recursively on  $[0, 1]$  by the following scheme: Suppose that  $Z_a^j$  is continuous and piecewise affine on each triadic interval  $[k3^{-j}, (k+1)3^{-j}]$ ,  $k \in \{0, \dots, 3^j - 1\}$ . Then  $Z_a^{j+1}$  is constructed as follows: On each triadic interval  $[k3^{-j}, (k+1)3^{-j}]$ ,  $Z_a^{j+1}$  is still a continuous function which is affine on each triadic subinterval  $[k'3^{-(j+1)}, (k'+1)3^{-(j+1)}]$  included in  $[k3^{-j}, (k+1)3^{-j}]$ , and

$$\begin{aligned} Z_a^{j+1}(k3^{-j}) &= Z_a^j(k3^{-j}) \\ Z_a^{j+1}(k3^{-j} + 3^{-(j+1)}) &= Z_a^j(k3^{-j}) + a \left( Z_a^j((k+1)3^{-j}) - Z_a^j(k3^{-j}) \right) \\ Z_a^{j+1}(k3^{-j} + 2 \cdot 3^{-(j+1)}) &= Z_a^j(k3^{-j}) + (1-a) \left( Z_a^j((k+1)3^{-j}) - Z_a^j(k3^{-j}) \right) \\ Z_a^{j+1}((k+1)3^{-j}) &= Z_a^j((k+1)3^{-j}). \end{aligned}$$

This simple construction is better explained by the Figure 6.

It is straightforward to see that the sequence  $(Z_a^j)_{j \geq 1}$  converges uniformly to a continuous function  $Z_a$  as soon as  $0 < a < 1$ . Bourbaki's function is obtained when  $a = 2/3$ , while Perkin's function corresponds to  $a = 5/6$ .

For  $a \leq 1/2$ , the function is simply the integral of a trinomial measure of parameters  $(a, 1-2a, a)$ , hence its singularity spectrum is completely known. We are going to explain why the functions  $Z_a$ , when  $a \geq 1/2$ , satisfy our assumptions, and thus can be written as the composition of a monofractal function  $g$  (with an exponent  $H$  we are going to determine) with an increasing function. We will also deduce from this study the singularity spectrum of  $Z_a$ .

For  $a > 1/2$ , the limit function  $Z_a$  is nowhere monotone. Let us compute the oscillations of  $Z_a$  on each triadic interval.

Remark first that the slope of  $Z_a^1$  on  $[0, 1/3]$  is  $3a$ , it is  $-3(2a-1)$  on  $[1/3, 2/3]$  and  $3a$  on  $[2/3, 1]$ . Iteratively, if  $j \geq 1$  and  $k \in \{0, \dots, 3^j - 1\}$ , we write  $k3^{-j} = \sum_{p=1}^j \xi_p 3^{-p}$ , with  $\xi_i \in \{0, 1, 2\}$ . Then the slope of  $Z_a^j$  on  $[k3^{-j}, (k+1)3^{-j}]$  is simply

$$(3a)^{n_{k,j,0}} (-3(2a-1))^{n_{k,j,1}} (3a)^{n_{k,j,2}} = 3^j (a)^{n_{k,j,0}} (-2a-1)^{n_{k,j,1}} (a)^{n_{k,j,2}},$$

where  $n_{k,j,i}$  is the number of integers  $p \in \{1, \dots, j\}$  such that  $\xi_p = i$  (for  $i = 0, 1, 2$ ) in the triadic decomposition of  $k3^{-j}$ .

Let us consider the trinomial measure  $\mu_a$  of parameters  $(\frac{a}{4a-1}, \frac{2a-1}{4a-1}, \frac{a}{4a-1})$ . Then it is obvious that the absolute value of the slope of  $Z_a^j$  on each triadic interval  $[k3^{-j}, (k+1)3^{-j}]$  can be written as  $\mu_a([k3^{-j}, (k+1)3^{-j}]) 3^j (4a-1)^j$ . As a final remark, we also notice that the oscillations of  $Z_a$  on each triadic interval  $[k3^{-j}, (k+1)3^{-j}]$  is the same as the oscillations of  $Z_a^j$  on each triadic interval  $[k3^{-j}, (k+1)3^{-j}]$ , which is equal to  $3^{-j}$  times the slope, i.e.

$$(38) \quad \mu_a([k3^{-j}, (k+1)3^{-j}]) (4a-1)^j.$$

Let  $q \in \mathbb{R}$ . Let us compute the sum of the oscillations of  $Z_a$  at generation  $j$ . We have

$$(39) \quad \sum_{k=0}^{3^j-1} (\omega_{[k3^{-j}, (k+1)3^{-j}]}(Z_a))^q = \sum_{k=0}^{3^j-1} (\mu_a([k3^{-j}, (k+1)3^{-j}]))^q 3^{jq \log_3(4a-1)}.$$

Let us explain now how we easily compute the exponent  $H_a$  such that (4) holds true. For a multinomial measure  $\mu_a$  (in fact, for any positive Borel measure), it is very classical in multifractal analysis to introduce the functions  $\tau_{\mu_a, j}(q)$  and the scaling function  $\tau_{\mu_a}(q)$  defined for  $q \in \mathbb{R}$  as (35) but in the triadic basis:

$$\tau_{\mu_a}(q) = \liminf_{j \rightarrow +\infty} \tau_{\mu_a, j}(q), \quad \text{where } \tau_{\mu_a, j}(q) = \frac{\log_3 \sum_{k=0}^{3^j-1} (\mu_a([k3^{-j}, (k+1)3^{-j}]))^q}{-j}.$$

In our simple case, it is easy to see that for every  $j \geq 1$  and  $q \in \mathbb{R}$

$$\begin{aligned} \tau_{\mu_a, j}(q) = \tau_{\mu_a}(q) &= -\log_3 \left( \left( \frac{a}{4a-1} \right)^q + \left( \frac{2a-1}{4a-1} \right)^q + \left( \frac{a}{4a-1} \right)^q \right) \\ &= -\log_3(2(a)^q + (2a-1)^q) + q \log_3(4a-1). \end{aligned}$$

What matters to us is the value of  $q$  for which the sum in (39) equals 1. Let us write this specific value  $q$  as  $1/H$ , for some  $H > 0$ . When this sum is 1, then we have

$$3^{-j \tau_{\mu_a}(1/H)} 3^{j(\log_3(4a-1))/H} = \sum_{k=0}^{3^j-1} (\mu_a([k3^{-j}, (k+1)3^{-j}]))^{1/H} 3^{j(\log_3(4a-1))/H} = 1.$$

Let  $H_a$  be the solution of the equation  $-\tau_{\mu_a}(1/H_a) + \log_3(4a-1)/H_a = 0$ , which is equivalent to

$$(40) \quad 2(a)^{1/H_a} + (2a-1)^{1/H_a} = 1.$$

This solution is positive, unique, and strictly smaller than 1. Hence, in this case, the monofractal exponent  $H_a$  is defined through an implicit formula.

In order to get the whole condition **C2**, it suffices to notice that any rescaled function  $(Z_a)_{J,K}$  (as defined in (6), but here with triadic intervals) is actually equal to  $Z_a$  (if  $Z_a$  is increasing on  $[k3^{-j}, (k+1)3^{-j}]$ ) or to  $Z_a(1-\cdot)$  (if  $Z_a$  is decreasing on  $[k3^{-j}, (k+1)3^{-j}]$ ).

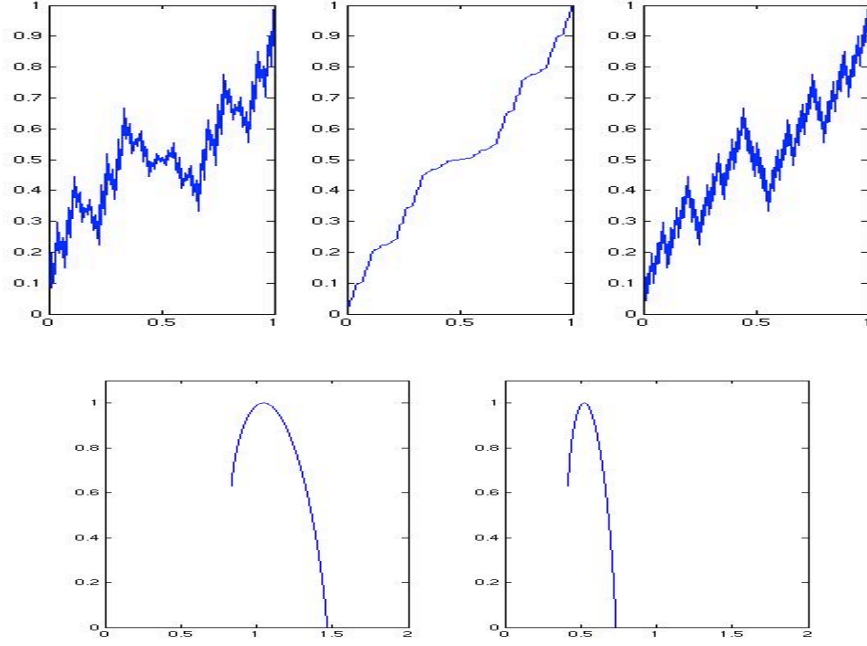


FIGURE 2. **Top:** Bourbaki's function  $f_{2/3}$  on the left, the multifractal time change  $F_{2/3}$  in the middle, and on the right the monofractal function  $g_{2/3}$  of exponent  $1/2$  such that  $f_{2/3} = g_{2/3} \circ F_{2/3}$ . **Bottom:** Singularity spectra of  $\mu_{2/3}$  on the left, of  $f_{2/3}$  on the right.

Hence  $H((Z_a)_{J,K})$  is a limit for every  $J, K$ , and is even constant equal to  $H_a$ . Thus **C2** is satisfied.

We can then apply Theorem 1.7, and  $Z_a$  is the composition of a monofractal function  $g_a$  of exponent  $H_a$  with an increasing function  $Z_a$ . For  $a = 2/3$ , we see that  $H_a = 1/2$  is the solution to (40), since  $2(2/3)^2 + (1/3)^2 = 1$ . We have plotted in Figure 6 the Bourbaki's function  $f_{2/3}$ , its corresponding time change  $F_{2/3}$  and the corresponding monofractal function  $g_{2/3}$  of exponent  $1/2$  such that  $f_{2/3} = g_{2/3} \circ F_{2/3}$ .

In this case, we can even go further and compute the singularity spectrum of  $Z_a$ . The trinomial measure satisfy the multifractal formalism for measures, i.e. the singularity spectrum of  $\mu$  is given by the Legendre transform of  $\tau_{\mu_a}$ :

$$d_{\mu_a}(\alpha) = (\tau_{\mu_a})^*(\alpha) := \inf_{q \in \mathbb{R}} (q\alpha - \tau_{\mu_a}(q)),$$

for every  $\alpha \in [-\log_3(a/(4a-1)), -\log_3(2a-1)/(4a-1)]$ .

It is easy to see, using (38), that if  $\mu_a$  has a local Hölder exponent equal to  $\alpha$  at a point  $t_0$ , then  $Z_a$  has at  $t_0$  a pointwise Hölder exponent equal to  $\alpha^{1/H_a} - \log_3(4a-1)$ . Hence the multifractal spectrum of  $Z_a$  is deduced from the one of  $\mu_a$  by the formula

$$d_{Z_a}(h) = d_{\mu_a} \left( (h + \log_3(4a-1))^{1/H} \right)$$



for every  $h \in [(-\log_3(a))^{1/H_a} - \log_3(4a-1), (-\log_3(2a-1))^{1/H_a} - \log_3(4a-1)]$ . A more explicit formula is obtained as follows: for every  $q \in \mathbb{R}$ , if  $\alpha = \tau_{\mu_a}$ , then  $d_{\mu_a}(\alpha) = q(\tau_{\mu_a})'(q) - \tau_{\mu_a}(q)$ .

The singularity spectra of  $f_{2/3}$  and  $\mu_{2/3}$  are given in Figure 6.

Finally, remark that the maximum of the spectrum is obtained for  $\alpha_a = ((\tau_{\mu_a})'(0))^{1/H_a} - \log_3(4a-1)$ , and  $d_{Z_a}(\alpha_a) = 1$ . After computations, we find  $\alpha_a = \frac{-1}{3}(\log_3(a^2(2a-1)))$ .

Let us consider the value of  $a_0$  such that  $\alpha_{a_0} = 1$ . Then  $a_0^2(2a_0-1) = 1/27$ , i.e.  $54a_0^3 - 27a_0^2 = 1$ . When  $a > a_0$ , the set of points  $t$  for which  $h_{Z_a}(t) > 1$  is of Lebesgue measure 1, hence we recover the main result of [15]:  $Z_a$  is differentiable on a set of Lebesgue measure 1. Here we obtain in addition the whole multifractal spectrum of  $Z_a$ .

## 7. RELATIONSHIP BETWEEN THE LOCAL EXPONENTS AND THE SPECTRA OF $f$ AND $Z$ WHEN $Z$ IS CMT

This last short Section is devoted to the possible transfer of the multifractal properties of  $f$  to the multifractal properties of  $\mu$ . We do not enter into much details, since this is not our main purpose in this work.

Assume that  $Z$  is CMT, i.e.  $Z = g \circ f$ , where  $g$  is monofractal with exponent  $0 < H < 1$  and  $f$  is the integral of a Borel measure  $\mu$  with support equal to  $[0, 1]$ : for every  $t \in [0, 1]$ ,  $f(t) = \mu([0, t])$ . Let us introduce the local upper exponents of

$$(41) \quad \bar{\alpha}_\mu(t) = \limsup_{r \rightarrow 0^+} \frac{\log \mu(B(t, r))}{\log r}.$$

This exponent is complementary to the exponent  $\alpha_\mu(t)$  defined in (9).

Let  $\varepsilon > 0$ . For a given  $t_0 \in [0, 1]$ , we obviously have when  $|\eta|$  is small enough

$$|Z(t+\eta) - Z(t)| = |g(f(t+\eta)) - g(f(t))| \leq |f(t+\eta) - f(t)|^{H-\varepsilon} \leq |\eta|^{(H-\varepsilon)(\alpha_\mu(t)-\varepsilon)}.$$

Hence  $h_Z(t) \geq H \cdot \alpha_\mu(t)$ .

The converse inequality does not hold in general, since the two inequalities above need not be satisfied simultaneously. Nevertheless, one always has

$$(42) \quad H \cdot \alpha_\mu(t) \leq h_Z(t) \leq H \cdot \bar{\alpha}_\mu(t).$$

One deduces from this inequality that

$$E_h^Z = \{t : h_Z(t) = h\} \subset \left( \bigcup_{h' \leq h/H} \{t \in [0, 1] : \alpha_\mu(t) = h'\} \right) \cap \left( \bigcup_{h' \geq h/H} \{t \in [0, 1] : \bar{\alpha}_\mu(t) = h'\} \right).$$

We denote respectively by  $F_{h/H}$  and  $\bar{F}_{h/H}$  the two unions of sets above.

From the above inequality, we only deduce that the comparison between the two multifractal spectra of  $Z$  and  $\mu$  is not straightforward in general! The case where  $\mu$  satisfies a multifractal formalism make things easier. Recall the  $L^q$ -spectrum (35) of  $\mu$ , and that  $\mu$  is said to obey the multifractal formalism [8, 6, 16] when the multifractal spectrum of  $\mu$  can be recovered as the Legendre transform of the scaling function of  $\mu$ , i.e.

$$d_\mu(h) = (\tau_\mu)^*(h) := \inf_{q \in \mathbb{R}} (qh - \tau_\mu(q)).$$

In most of the classical cases (random and deterministic Gibbs measures, self-similar measures as in Section 5, Mandelbrot cascades, cookie-cutters,...), the Hausdorff dimension

$d_\mu(h)$  is obtained by points for which the local exponent of the measure  $\mu$  is obtained as a limit (and not only as a liminf), and thus

$$\text{for every } h' \geq 0, \dim E_{h'}^\mu = \dim \{t : \alpha_\mu(t) = \bar{\alpha}_\mu(t) = h'\}.$$

Recalling (42), we see that

$$\dim E_{h/H}^\mu \leq \dim E_h^Z \leq \min(\dim F_{h/H}, \dim \bar{F}_{h/H}).$$

In order to conclude, we use two properties following from the verification of the multifractal formalism (again, see [8, 6, 16]):

- In the increasing part of the multifractal spectrum of  $\mu$  (when  $h' \leq \tau'_\mu(0^+)$ ), one always has  $\dim(F_{h'}) = \dim E_{h'}^\mu$ ,
- In the decreasing part of the multifractal spectrum of  $\mu$  (when  $h' \geq \tau'_\mu(0^+)$ ), one always has  $\dim(\bar{F}_{h'}) = \dim \bar{E}_{h'/H}^\mu = \dim E_{h'/H}^\mu$ .

This yields  $\dim E_{h/H}^\mu = \dim E_h^Z$  for every  $h \geq 0$ , hence  $d_Z(h) = d_\mu(h/H)$ .

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