LOCAL BEHAVIOR OF TRACES OF BESOV FUNCTIONS: PREVALENT RESULTS

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ABSTRACT. Let $1 \leq d < D$ and (p, q, s) satisfying $0 < p < \infty$, $0 < q \leq \infty$, $0 < s - d/p < \infty$. In this article we study the global and local regularity properties of traces, on affine subsets of $\mathbb{R}^D$, of functions belonging to the Besov space $B^s_{p,q}(\mathbb{R}^D)$. Given a d-dimensional subspace $\mathcal{H} \subset \mathbb{R}^D$, for almost all functions in $B^s_{p,q}(\mathbb{R}^D)$ (in the sense of prevalence), we are able to compute the singularity spectrum of the traces f_a of f on affine subspaces of the form $a + \mathcal{H}$, for Lebesgue-almost every $a \in \mathbb{R}^{D-d}$. In particular, we prove that for Lebesgue-almost every $a \in \mathbb{R}^{D-d}$, these traces f_a are more regular than what could be expected from standard trace theorems, and that f_a enjoys a multifractal behavior.

1. Introduction

Investigating regularity properties of traces of functions belonging to some Besov or Sobolev spaces is a longstanding issue. For instance, such questions arise from PDE's theory, where the Dirichlet condition imposes some regularity properties of the trace of the solution on the frontier of the domain. In this article, we study the local behavior of traces of functions belonging to the Besov space $B^s_{p,q}(\mathbb{R}^D)$ on d-dimensional affine subspaces of $\mathbb{R}^D$.

Not only concerned with global smoothness properties (i.e. to which Sobolev and Besov spaces the traces belong), we will especially focus on the local behavior of such traces. The notion of pointwise regularity we discuss in the sequel is the following. Given a real function $f \in L^\infty_{\text{loc}}(\mathbb{R}^D)$ and $x_0 \in \mathbb{R}^D$, f is said to belong to $\mathcal{C}^\alpha(x_0)$, for some $\alpha \geq 0$, if there exists a polynomial P of degree at most $\lfloor \alpha \rfloor$ and a constant $C > 0$ such that locally around x_0 :

$$|f(x) - P(x - x_0)| \leq C|x - x_0|^\alpha. \tag{1}$$

The local regularity of f at x_0 is measured by the *pointwise Hölder exponent* :

$$h_f(x_0) = \sup\{\alpha \geq 0 : \ f \in \mathcal{C}^\alpha(x_0)\}.$$

As will be observed soon, this exponent $h_f(x_0)$ may vary rather erratically with x_0, and the relevant information is then provided by the *spectrum of singularities* d_f of f, which is the function :

$$d_f : h \in [0, \infty] \longmapsto \dim_{\mathcal{H}} E_f(h), \quad \text{where } E_f(h) := \{x_0 \in \mathbb{R}^D : h_f(x_0) = h\}.$$

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Here $\dim_{\mathcal{H}}$ stands for the Hausdorff dimension. We adopt the convention that $\dim_{\mathcal{H}} \emptyset = -\infty$. The spectrum of singularities d_f describes the geometrical repartition of the singularities of f.

This spectrum and its relevance in physics, especially in fluid mechanics, goes back to the 1980's. At this time, physicists have been able to measure the velocity of a turbulent fluid along one direction, and they observed that their signals exhibited very different local behaviors at different times. This variability was proposed by Frisch and Parisi as a possible explanation for the concavity of the scaling function associated with the velocity (see [8] and several references on the subject). These works are very intimately related to our questions, since only the trace of the fluid's velocity is measured in practice. Hence, to infer some results on the regularity properties of the three-dimensional velocity, it is key to investigate the possible local behavior of traces of Sobolev or Besov functions.

Precise results on the pointwise regularity of functions belonging to classical spaces such as Besov $B^s_{p,\infty}(\mathbb{R}^D)$ spaces have recently been obtained [1, 7, 12, 13]. These results are of two kinds: universal upper bounds for the spectrum of singularities (valid for any element of the space) and almost-sure spectrum (valid for a "large" subset of the space, in the sense of prevalence or Baire categories). We detail these results, as well as ours, now.

In all that follows, $0 < d < D$ are two fixed integers. Let $d' := D - d$ and $(x, x') \in \mathbb{R}^d \times \mathbb{R}^{d'} = \mathbb{R}^D$. For $a \in \mathbb{R}^{d'}$ we shall denote by $\mathcal{H}_a := \{(x, a)\}$ the d-dimensional affine subspace of $\mathbb{R}^D$.

Let f be a continuous function on $\mathbb{R}^D$. Its trace on $\mathcal{H}_a$ is

$$f_a := f_{|\mathcal{H}_a} : \begin{array}{rcl} \mathbb{R}^d & \longrightarrow & \mathbb{R} \\ x & \longmapsto & f(x,a) \end{array}$$

If f is not continuous, its trace can be defined by Fourier regularization: we shall again write f_a for $\lim_{N\to\infty} \big(\mathcal{F}^{-1}\big(\mathbf{1}_{|\xi|\le N}\mathcal{F}f\big)\big)_a$, whenever that limit exists.

Standard trace theorems inevitably involve a loss of regularity, for instance it is well known that when $s > 1/2$, the trace of $f \in H^s(\mathbb{R}^2)$ on any one-dimensional subspace belongs to $H^{s-1/2}(\mathbb{R})$. Similar results hold for Besov spaces (see § 2.4): it can easily be shown that the trace operator $f \mapsto f_a$ maps $B^s_{p,\infty}(\mathbb{R}^D)$ to $B^{s-d'/p}_{p,\infty}(\mathbb{R}^d)$. However, most of the traces f_a have better properties than expected. Indeed, we will prove the following theorem:

Theorem 1.1. *Let $0 < p, s < \infty$. If $f \in B^s_{p,\infty}(\mathbb{R}^D)$, then for Lebesgue-almost all $a \in \mathbb{R}^{d'}$ $f_a \in \bigcap_{s'<s} B^{s'}_{p,\infty}(\mathbb{R}^d)$.*

In particular, when $s > d/p$, the inclusion $B^{s-\varepsilon}_{p,\infty}(\mathbb{R}^d) \hookrightarrow C^{s-\varepsilon-d/p}(\mathbb{R}^d)$ for any $\varepsilon > 0$ small enough implies that Lebesgue-almost all traces f_a exist and are uniform Hölder functions.

A version of Theorem 1.1 was proved by Jaffard in [11]. We present a short and independent proof of it in Section 3, which implies a stronger result than that of [11]. Indeed, we prove that the loss of Besov regularity between f and Lebesgue-almost all traces f_a is of logarithmic order (see equation (23) in Section 3).

By another result of Jaffard [12], belonging to a Besov space yields an upper bound on the spectrum of singularities:

Theorem 1.2. *Let $0 < p < \infty$ and $d/p < s < \infty$. For any $g \in B^s_{p,\infty}(\mathbb{R}^d)$, for all $h \geq s - d/p$,*

$$d_g(h) \leq \min(d, d + (h - s)p),$$

and $E_f(h) = \emptyset$ if $h < s - d/p$.

Remark 1.3. *The results so far have been stated for Besov spaces $B^s_{p,q}$ with $q = \infty$ but it is clear from classical Besov embeddings (see equation* (10) *below) that they hold identically for any $q > 0$.*

Not only is Theorem 1.2 optimal, the upper bound is actually an *almost sure* equality in $B^s_{p,q}(\mathbb{R}^D)$ (Theorem 1.4) in the sense of prevalence, as explained below.

Prevalence theory is used to supersede the notion of Lebesgue measure in any real or complex topological vector space E. This notion was proposed by Christensen [4] and independently by Hunt *et al.* [9]. The space E is endowed with its Borel σ-algebra $\mathcal{B}(E)$ and all Borel measures μ on $(E, \mathcal{B}(E))$ will be automatically *completed*, that is we put $\mu(A) := \mu(B)$ if $B \in \mathcal{B}(E)$ and the symmetric difference $A \Delta B$ is included in some $D \in \mathcal{B}(E)$ with $\mu(D) = 0$. A set is said to be *universally measurable* if it is measurable for any (completed) Borel measure.

Definition 1. *A universally measurable set $A \subset E$ is called* shy *if there exists a Borel measure μ that is positive on some compact subset K of E and such that*

$$\text{for every } x \in E, \quad \mu(A + x) = 0.$$

More generally, a set that is included in a shy universally measurable set is also called shy.

Finally, the complement in E of a shy subset is called prevalent.

The measure μ used to show that some subset is shy or prevalent is called a *probe*. It can be for instance the Lebesgue measure carried by some finite-dimensional subspace of E: this is the technique that will be used in § 5.2.

When a set B is prevalent, it is dense in E, $B + x$ is also prevalent for any $x \in E$ and if $(B_n)_{n \in \mathbb{N}}$ is a sequence of prevalent sets then so is $\bigcap_{n \in \mathbb{N}} B_n$. Finally, when E has finite dimension, B is prevalent in E if and only if it has full Lebesgue measure. This justifies that a prevalent set B is a "large" set in E and extends reasonably the notion of full Lebesgue measure to infinite dimensional spaces.

From now on, without any possible confusion, the term "almost all" will be indiscriminately used to describe elements in a prevalent subset of an infinite-dimensional space, or in a subset having full Lebesgue measure in a finite-dimensional space.

The use of prevalence in function spaces was pioneered by Hunt [10]. Further developing the technique, Jaffard and Fraysse [7] proved the following:

Theorem 1.4. *Let $0 < p < \infty$, $0 < q \le \infty$ and $0 < s - D/p < \infty$. For almost all $g \in B^s_{p,q}(\mathbb{R}^D)$,*

$$d_g(h) = \begin{cases} D + (h-s)p & \text{if } h \in [s - D/p, s] \\ -\infty & \text{else} \end{cases}$$

and for x in a set of full Lebesgue measure in $\mathbb{R}^D$, $h_g(x) = s$.

Remark 1.5. *Another notion of genericity is given by Baire's theory: a property is said to be quasi-sure in a complete metric space E if this property is realized on a residual (comeagre) set in E. We choose to work within the prevalence framework, but Baire's genericity is also worthy of interest and will be studied in a subsequent paper.*

In this paper we prove the following result on the singularity spectrum of traces of almost all Besov functions.

Theorem 1.6. *Let $0 < p < \infty$, $0 < q \le \infty$ and $0 < s - d/p < +\infty$. For almost all f in $B^s_{p,q}(\mathbb{R}^D)$, for Lebesgue-almost all $a \in \mathbb{R}^{d'}$, the following holds:*

(i) the spectrum of singularities of f_a is

$$d_{f_a}(h) = \begin{cases} d + (h-s)p & \text{if } h \in [s - d/p, s] \\ -\infty & \text{else.} \end{cases} \tag{2}$$

(ii) for every open set $\Omega \subset \mathbb{R}^d$, the level set $E_{f_a}(s) \cap \Omega$ has full Lebesgue measure in Ω.

Let us make some remarks on Theorem 1.6:

• In a given Besov space $B^s_{p,q}(\mathbb{R}^D)$ (Theorem 1.4), as well as in $C^\alpha(\mathbb{R}^D)$ [14] or for Borel measures supported by $[0,1]^D$ [2], the almost-sure regularity is often the "worst possible", i.e. the upper bound on the spectrum valid for all elements of the considered space turns out to be an equality for almost all functions or measures. This is not the case in Theorem 1.6, for which the almost sure spectrum does not coincide with the *a priori* upper bound, and thus the traces are more regular than what could be expected *a priori*.

• Compared to [10] and [7], the consideration of traces implies that the prevalent set can only be indirectly defined, which makes the question of its (universal) measurability nontrivial, especially in the nonseparable case.

• Observe that the singularities with Hölder exponent h less than $s - d/p$ are "not seen" by Lebesgue-almost every traces f_a. This corresponds to the level sets $E_f(h)$ of Hausdorff dimension less than d'. B. Mandelbrot referred to this phenomenon as *negative dimensions*: By this, he means that almost every function $f \in B^s_{p,q}(\mathbb{R}^D)$ possesses singularities with exponent $s - D/p \le h < s - d/p$, but these singularities form a set of too small a dimension to intersect a large quantity among the hyperplanes $\mathcal{H}_a$ of dimension $d' = D - d$.

Remark 1.7. *Theorem 1.6 and the above remark are reminiscent of classical results of P. Mattila* [16] *on the Hausdorff dimensions of intersection of fractal subset of $\mathbb{R}^D$ with Lebesgue-almost all d-dimensional hyperplanes, or of sliced measures* [16, 15].

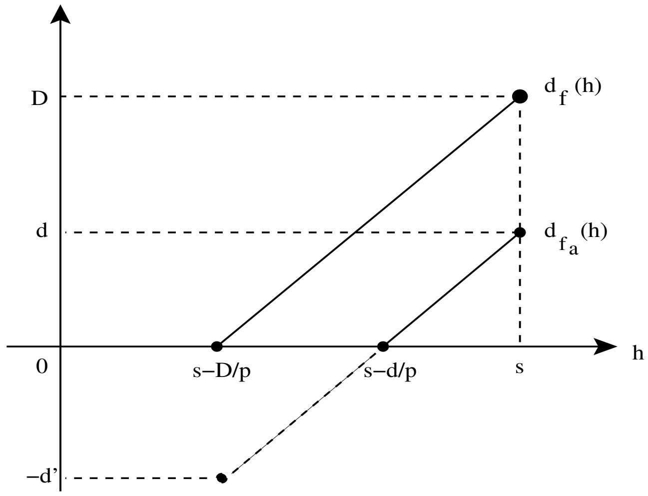

FIGURE 1. Singularity spectrum of almost all $f \in B^s_{p,q}(\mathbb{R}^D)$ and its trace f_a for Lebesgue almost every $a \in \mathbb{R}^{d'}$.

In this theorem, all the hyperplanes $\mathcal{H}_a$ on which the traces are taken are parallel (to the d first coordinates axes). Since the Besov spaces are invariant by unitary transformation of the coordinates, the result remains valid in any fixed direction. Thanks to the stability of prevalence by countable intersection, we thus obtain:

Corollary 1.8. *Let Δ be a countable subset of the Grassmannian $\mathrm{Gr}_d(D)$. Under the same hypotheses on p, q, s, for almost all f in $B^s_{p,q}(\mathbb{R}^D)$, for any $\mathcal{H} \in \Delta$, for Lebesgue-almost all $a \in \mathcal{H}^\perp$, the trace of f on $\mathcal{H} + a$ has the properties stated in Theorem 1.6.*

Unfortunately no Fubini theorem holds for prevalence, so we cannot directly deduce from this the natural generalization below, which we leave for subsequent studies.

Conjecture 1.9. *Consider the Grassmannian $\mathrm{Gr}_d(D)$ and its Haar measure $\mu_{d,D}$. For almost all f in $B^s_{p,q}(\mathbb{R}^D)$, for $\mu_{d,D}$-almost all $\mathcal{H} \in \mathrm{Gr}_d(D)$, for Lebesgue-almost all $a \in \mathcal{H}^\perp$, the trace of f on $\mathcal{H}+a$ has the properties stated in Theorem 1.6.*

The paper is organized as follows. Our method is based on wavelets, and requires various notions of real and functional analysis. Section 2 provides all the definitions and important results needed to complete the proof of Theorem 2.1. In Section 4, we prove the upper bound for the singularity spectrum for all functions in $B^s_{p,q}(\mathbb{R}^D)$, and the lower bound for all functions in a set that we call $\mathcal{F}$. Then, in Section 5, we show that this set $\mathcal{F}$ is prevalent, the main difficulties lying in the measurability properties of $\mathcal{F}$. Appendix **??** contains a shorter proof of Theorem 1.1, and Appendix A deals with the universal measurability of $\mathcal{F}$ in the case $q = +\infty$ (which differs from the case $q < +\infty$ since $B^s_{p,\infty}(\mathbb{R}^D)$ is not separable).

2. Preliminaries

2.1. **Dimensions.**

Two notions of dimensions of sets in $\mathbb{R}^d$ will be used below: the Hausdorff dimension and the upper box dimension. We recall them quickly.

Let E be a bounded set in $\mathbb{R}^d$. For every $\varepsilon > 0$, denote by $N_\varepsilon(E)$ the minimal number of cubes of size ε needed to cover the set E. The upper box dimension of E, denoted by $\overline{\dim_B}(E)$, is the real number $\in [0,d]$ defined as

$$\overline{\dim_B}(E) = \limsup_{\varepsilon\to 0^+} \frac{\log N_\varepsilon(E)}{-\log \varepsilon}. \tag{3}$$

For the reader's convenience we also recall the definition of the Hausdorff dimension.

Definition 2. *Let $s \geq 0$. The s-dimensional Hausdorff measure of a set E, $\mathcal{H}^s(E)$, is defined as*

$$\mathcal{H}^s(E) = \lim_{r\searrow 0} \mathcal{H}^s_r(E), \quad \textit{with } \mathcal{H}^s_r(E) = \inf\Big\{\sum_i |E_i|^s\Big\},$$

the infimum being taken over all the countable families of sets E_i such that $|E_i| \leq r$ and $E \subset \bigcup_i E_i$. Then, the Hausdorff dimension of E, $\dim_{\mathcal{H}} E$, is defined as

$$\dim_{\mathcal{H}} E = \inf\{s \geq 0 : \mathcal{H}^s(E) = 0\} = \sup\{s \geq 0 : \mathcal{H}^s(E) = +\infty\}.$$

For a bounded set $E \subset \mathbb{R}^d$, we have

$$0 \leq \dim_{\mathcal{H}}(E) \leq \overline{\dim_B}(E) \leq d.$$

2.2. **Wavelets.**

We recall very briefly the basics of multiresolution wavelet analysis (for details see for instance [5]). For an arbitrary integer $N \geq 1$ one can construct compactly supported functions $\Psi^0 \in C^N(\mathbb{R})$ (called the scaling function) and $\Psi^1 \in C^N(\mathbb{R})$ (called the mother wavelet), with Ψ^1 having at least $N+1$ vanishing moments (i.e. $\int_{\mathbb{R}} x^n \Psi^1(x)dx = 0$ for $n \in \{0,\ldots,N\}$), and such that the set of functions

$$\Psi^1_{j,k} : x \mapsto \Psi^1(2^j x - k)$$

for $j \in \mathbb{Z}, k \in \mathbb{Z}$ form an orthogonal basis of $L^2(\mathbb{R})$ (note that we choose the L^∞ normalization, not L^2). In this case, the wavelet is said to be N-regular.

Let us introduce the notations

$$0^d := (0,0,\cdots,0), \qquad 1^d := (1,1,\cdots,1), \qquad L^d := \{0,1\}^d \backslash 0^d.$$

An orthogonal basis of $L^2(\mathbb{R}^d)$ is then obtained by tensorization. For every $\lambda := (j,\mathbf{k},\mathbf{l}) \in \mathbb{Z}\times\mathbb{Z}^d\times\{0,1\}^d$, let us define the tensorized wavelet

$$\Psi_\lambda(x) := \prod_{i=1}^d \Psi^{l_i}_{j,k_i}(x_i),$$

with obvious notations that $\mathbf{k} = (k_1,k_2,\cdots,k_d)$ and $\mathbf{l} = (l_1,l_2,\cdots,l_d)$.

Any function $f \in L^2(\mathbb{R}^d)$ can be written (the inequality being true in $L^2(\mathbb{R}^d)$)

$$f = \sum_{\lambda=(j,\mathbf{k},\mathbf{l}):\, j\in\mathbb{Z},\, \mathbf{k}\in\mathbb{Z}^d,\, \mathbf{l}\in L^d} c_\lambda \Psi_\lambda(x), \tag{4}$$

where

$$c_\lambda := 2^{jd} \int_{R^d} f(x)\Psi_\lambda(x)\, dx. \tag{5}$$

It is implicit in (5) that the wavelet coefficients depend on f. Observe that in the wavelet decomposition (4), no wavelet Ψ_λ such that $\mathbf{l} = 0^d$ (where $\lambda = (j, \mathbf{k}, \mathbf{l})$) appears.

Similar notations (e.g. $\lambda^D := (j, (\mathbf{k}, \mathbf{k}'), (\mathbf{l}, \mathbf{l}')) \in \mathbb{Z}\times\mathbb{Z}^D\times\{0,1\}^D$) with the straightforward modifications will produce an orthogonal basis of $L^2(\mathbb{R}^D)$. The wavelets and the corresponding wavelet coefficients in $L^2(\mathbb{R}^D)$ will be denoted respectively by Ψ_{λ^D} and c_{λ^D}.

In 5.2 we shall need to consider the 1-periodic function

$$G : t \in \mathbb{R} \longmapsto \sum_{k\in\mathbb{Z}} \Psi^1(t-k). \tag{6}$$

and make the technical hypothesis on Ψ^1:

$$(\mathcal{H}_N) \qquad \begin{cases} \text{(i)} & \Psi^1 \text{ is } N\text{-regular,} \\ \text{(ii)} & \text{The set } Z := G^{-1}(\{0\}) \cap [0,1] \text{ is finite,} \\ \text{(iii)} & \text{For every } t \in Z,\ |G'(t)| > 0\ . \end{cases}$$

This condition is very reasonable for a given wavelet Ψ^1. Numerical simulations (see figure 2) indicate that $(\mathcal{H}_N)$ is verified for suitable choices of regular wavelets, including in particular Daubechies's compactly supported wavelets [5]. In Figure 2, the simulations of Ψ^1 and $(\Psi^1)'$ (computed using the associated wavelet filters) are precise enough to guarantee that G' does not vanish around the zeros of G.

2.3. **Localization of the problem.**

We will be first focusing on the local behavior of traces on $(0,1)^d\times\{a\}$, $a \in (0,1)^{d'}$. As Proposition 2.3 shows, if f is written as (4), only the coefficients c^D_λ such that $j \geq 0$ and $(\mathbf{k}2^{-j}, \mathbf{k}'2^{-j}) \in [0,1]^D$ can play a role in the value of the pointwise exponent $h_{f_a}(x)$. For our purpose, we can identify functions that have the same wavelet coefficients c^D_λ when $(\mathbf{k}2^{-j}, \mathbf{k}'2^{-j}) \in [0,1]^D$. Hence we will consider functions f of the form

$$f = \sum_{\lambda^D \in \Lambda^D \times L^D} c_{\lambda^D} \Psi_{\lambda^D}(x), \tag{7}$$

where

$$\begin{aligned} \text{for } j \geq 1, \quad \mathbb{Z}_j &= \{0, 1, \cdots, 2^j - 1\} \quad \text{and} \quad \Lambda^D_j = \{j\} \times \mathbb{Z}^D_j \\ \Lambda^D &= \bigcup_{j\geq 1} \Lambda^D_j. \end{aligned}$$

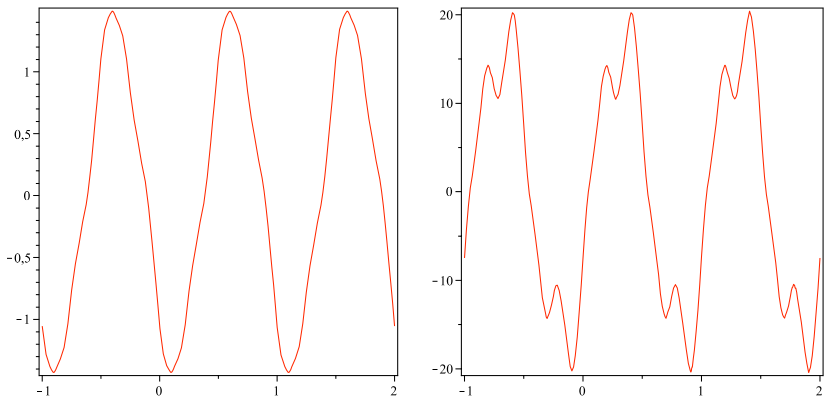

FIGURE 2. Plot of the periodized Daubechies wavelet with eight vanishing moments, and its derivative. The approximation is precise enough to ensure that G and its derivative do not vanish at the same time.

If we prove Theorem 1.6 on $[0,1]^D$ instead of $\mathbb{R}^D$, then by dilation it will be true on any cube $[-N,N]^D$. Prevalence results being stable by countable intersection on $N \in \mathbb{N}$, Theorem 1.6 will thus be obtained.

We shall present our results in this framework, and we will effectively prove the following:

Theorem 2.1. *Let* $0 < p < \infty$, $0 < q \leq \infty$. *Assuming that* $s > d/p$, *for almost all* f *in* $B^s_{p,q}([0,1]^D)$, *for almost all* $a \in [0,1]^{d'}$, *the following holds:*

(i) the spectrum of singularities of f *is*

$$d_{f_a}(h) = \begin{cases} d + (h-s)p & \text{if } h \in [s-d/p, s] \\ -\infty & \text{else.} \end{cases} \tag{8}$$

(ii) the level set $E_{f_a}(s)$ *has full Lebesgue measure in* $[0,1]^d$.

2.4. Characterization of local and global regularity properties.

Let $0 < s < \infty$, $0 < p,q \leq \infty$. Assume that the wavelet Ψ is at least $[s+1]$-regular. The $B^s_{p,q}([0,1]^D)$ Besov norm (quasi-norm when $p < 1$ or $q < 1$) of a function f on $[0,1]^D$ having wavelet coefficients c_{λ^D} is defined as

$$\|f\|_{B^s_{p,q}} = \left(\sum_{j \geq 1} \left(2^{(sp-D)j} \sum_{(\mathbf{k},\mathbf{k}') \in \mathbb{Z}^D_j} |c_{\lambda^D}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \tag{9}$$

with the obvious modifications when $p = \infty$ or $q = \infty$. The Besov space $B^s_{p,q}([0,1]^D)$ is naturally the set of functions with finite (quasi-)norm. It is a complete metrizable space, normed when p and $q \geq 1$, separable when both are finite.

The following standard embeddings are easy to deduce from (9): For any $0 < s < \infty$, $0 < p \leq \infty$, $0 < q < q' \leq \infty$, $\varepsilon > 0$,

$$B^s_{p,q}([0,1]^D) \hookrightarrow B^s_{p,q'}([0,1]^D) \hookrightarrow B^{s-\varepsilon}_{p,q}([0,1]^D) \tag{10}$$

Remark 2.2. *In contrast with Theorems 1.1 and 1.2, the prevalence result for a given $q < \infty$ cannot simply be deduced from the result for $q = \infty$ (nor the other way round). Indeed it can be shown that in (10) each included space is shy in the next one.*

Let us finally recall the fundamental result linking pointwise regularity and the size of wavelet coefficients, which justifies our approach.

Proposition 2.3. *Suppose that $\gamma > 0$ and the wavelet Ψ is at least $[\gamma + 1]$-regular. Let $f : [0,1]^d \to \mathbb{R}$ be a locally bounded function with wavelet coefficients $\{c_\lambda\}$, and let $x \in [0,1]^d$.*

If $f \in C^\gamma(x)$, then there exists a constant $M < \infty$ such that for all $\lambda = (j, \mathbf{k}, \mathbf{l}) \in \Lambda^d \times L^d$,

$$|c_\lambda| \leq M\big(2^{-j} + \big|x - k2^{-j}\big|\big)^\gamma = M2^{-j\gamma}(1 + \big|2^j x - k\big|)^\gamma \tag{11}$$

Reciprocally, if (11) holds true and if $f \in \bigcup_{\varepsilon>0} C^\varepsilon([0,1]^d)$, then $f \in C^{\gamma-\eta}(x)$, for every $\eta > 0$.

Finally, the notion of cone of influence will be needed later.

Definition 3. *Let $L > 0$. The cone of influence of width L above $x \in \mathbb{R}^d$ is the set of cubes $(j, \boldsymbol{k}, \mathbf{l}) \in \Lambda^d$ such that*

$$|x - \boldsymbol{k}2^{-j}| \leq L2^{-j}.$$

2.5. **Traces.**

Recall that for $a \in [0,1]^{d'}$ and f continuous on $[0,1]^D$, the function f_a is simply defined as $f_a(x) := f(x,a)$. Moreover, recall that $\lambda = (j, \mathbf{k}, \mathbf{l})$ with $j \in \mathbb{N}^*$, $\mathbf{k} \in \mathbb{Z}^d_j$ and $\mathbf{l} \in \{0,1\}^d$ and that $\lambda^D = (j, (\mathbf{k}, \mathbf{k}'), (\mathbf{l}, \mathbf{l}'))$ with $j \in \mathbb{N}^*$, $\mathbf{k} \in \mathbb{Z}^d_j$, $\mathbf{k}' \in \mathbb{Z}^{d'}_j$, $\mathbf{l} \in \{0,1\}^d$ and $\mathbf{l}' \in \{0,1\}^{d'}$. Using the expansion (7) of f in the tensorized wavelet basis $\{\Psi_{\lambda^D}\}$, we have

$$\begin{aligned} f_a(x) &= \sum_{\lambda^D \in \Lambda^D \times L^D} c_{\lambda^D} \prod_{i=1}^{d} \Psi^{l_i}_{j,k_i}(x_i) \prod_{i=1}^{d'} \Psi^{l'_i}_{j,k'_i}(a_i) \\ &= G_a(x) + F_a(x) \end{aligned} \tag{12}$$

where

$$G_a(x) := \sum_{\lambda \in \Lambda^d \times 0^d} d_\lambda(a) \Psi_\lambda(x) \tag{13}$$

$$F_a(x) := \sum_{\lambda \in \Lambda^d \times L^d} d_\lambda(a) \Psi_\lambda(x) \tag{14}$$

and for $\lambda = (j, \mathbf{k}, \mathbf{l}) \in \Lambda^d \times \{0,1\}^d$,

$$\text{if } \mathbf{l} = 0^d, \quad d_\lambda(a) := \sum_{\substack{\lambda^D = (j,(\mathbf{k},\mathbf{k}'),(0^d,\mathbf{l}')): \\ \mathbf{k}' \in \mathbb{Z}^{d'}_j, \mathbf{l}' \in L^{d'}}} c_{\lambda^D} \prod_{i=1}^{d'} \Psi^{l'_i}_{j,k'_i}(a_i). \tag{15}$$

$$\text{if } \mathbf{l} \in L^d, \quad d_\lambda(a) := \sum_{\substack{\lambda^D = (j,(\mathbf{k},\mathbf{k}'),(\mathbf{l},\mathbf{l}')): \\ \mathbf{k}' \in \mathbb{Z}^{d'}_j, \mathbf{l}' \in \{0,1\}^{d'}}} c_{\lambda^D} \prod_{i=1}^{d'} \Psi^{l'_i}_{j,k'_i}(a_i). \tag{16}$$

Formula (14) indeed yields a wavelet decomposition of the function F_a, since the wavelets appearing in (14) form a wavelet basis of $L^2([0,1]^d)$ (if completed by the function $\Psi_{0,0^d,0^d}$). This is not the case for the function G_a with formula (13), since only the scaling function Ψ^0 is used in this decomposition. Fortunately, we have the following standard result for the Besov properties of a function G_a defined through a formula like (13).

Proposition 2.4. *If $s_0 > 0$, and $g(x) = \sum_{\lambda\in\Lambda^d\times 0^d} d_\lambda \Psi_\lambda(x)$ with $\{d_\lambda\}$ satisfying*

$$\sup_{j\geq 1} \; 2^{j(p_0 s_0 - d)} \left(\sum_{\lambda=(j,\mathbf{k},\mathbf{l}):\, \mathbf{k}\in\mathbb{Z}_j^d,\, \mathbf{l}=0^d} |d_\lambda|^p \right) < +\infty, \tag{17}$$

then $g \in B^{s_0}_{p_0,\infty}([0,1]^d)$.

Proposition 2.4 entails that the same Besov characterization as (9) when one considers only scaling functions. The proof of Proposition 2.4, that we do not reproduce here, consists of decomposing each scaling function Ψ_λ, for $\lambda = (j,\mathbf{k},0^d)$ on the wavelets of smaller frequencies, i.e. on $\Psi_{\widetilde{\lambda}}$ with $\widetilde{\lambda} = (\widetilde{j},\widetilde{\mathbf{k}},\widetilde{\mathbf{l}})$ such that $\widetilde{j} \leq j$ and $\widetilde{\mathbf{l}} \in L^d$.

As a conclusion, the trace f_a can be written

$$f_a = \sum_{\lambda\in\Lambda^d\times\{0,1\}^d} d_\lambda(a)\Psi_\lambda(x) \tag{18}$$

where for $\lambda = (j,\mathbf{k},\mathbf{l}) \in \Lambda^d \times \{0,1\}^d$, $d_\lambda(a)$ is given by (15) and (16). For such a decomposition, the Besov characterization (9) holds true, the difference with (4) is that the sum over $\lambda \in L^d$ is replaced by $\lambda \in \{0,1\}^d$.

Recalling now Theorem 1.1 (proved in Section 3), $f_a \in \bigcap_{\varepsilon>0} B^{s-\varepsilon}_{p,\infty}([0,1]^d)$ for Lebesgue-almost every $a \in [0,1]^d$. Hence, still for almost every a, we can consider the *effective* wavelet decomposition of f_a on the wavelet basis provided by (4), and we write

$$f_a = c_{0,0^d,0^d}\Psi_{0,0^d,0^d}(x) + \sum_{\lambda\in\Lambda^d\times L^d} c_\lambda(a)\Psi_\lambda(x). \tag{19}$$

We will use both forms (18) and (19).

2.6. **Dyadic approximation.**

Let $B(x,r)$ denote the closed l^∞ ball of radius r around x in $[0,1]^d$. For $\alpha \geq 1$ and $j \in \mathbb{N}$, let

$$\begin{aligned} \mathcal{X}_j^\alpha &:= \bigcup_{k\in\mathbb{Z}_j^d} B(k2^{-j}, 2^{-j\alpha}) \\ \text{and} \qquad \mathcal{X}^\alpha &:= \limsup_{j\to\infty} \mathcal{X}_j^\alpha \end{aligned}$$

The set $\mathcal{X}^\alpha$ is constituted by points in $[0,1]^d$ that are approached at rate at least α by dyadics. In other words, $x \in \mathcal{X}^\alpha$ if and only if there exists a sequence $(J_n, K_n)_{n\geq 1} \in \Lambda^d$ such that $J_n \to +\infty$ and for all $n \in \mathbb{N}$

$$\left| x - K_n 2^{-J_n} \right| \leq 2^{-\alpha J_n}. \tag{20}$$

Observe that $\mathcal{X}^1 = [0,1]^d$ and if $\alpha \leq \alpha'$ then $\mathcal{X}^{\alpha'} \subset \mathcal{X}^\alpha$. Observe also that if $x \in \mathcal{X}^\alpha$ is not itself a dyadic, then the sequence (J_n, K_n) can be chosen so that for every n the fraction $\frac{\mathbf{K}_n}{2^{J_n}}$ is irreducible. We call $(J_n, K_n)_{n\geq 1}$ an *irreducible* sequence.

About the dimension of $\mathcal{X}^\alpha$, a well know result (for instance proved in [6]) states:

Theorem 2.5. *There exists a positive σ-finite measure m_α carried by $\mathcal{X}^\alpha$ and such that any set E having Hausdorff dimension $\dim_{\mathcal{H}}(E) < \frac{d}{\alpha}$ has measure $m_\alpha(E) = 0$.*

In particular, $m_\alpha(\mathcal{X}^\alpha) > 0$ and $\dim_{\mathcal{H}} \mathcal{X}^\alpha = d/\alpha$.

2.7. **Prevalence, universal measurability, analytic sets.**

In the Definition 1 of the prevalence in a complete metric space E, a set $B \subset E$ needs to be universally measurable to be shy or prevalent (this includes the Borel sets). One main difficulty occurring in the proof of Theorem 1.6 lies in the universal measurability property of subsets of E for which we aim to prove a prevalence property. Indeed, these sets will be defined through complicated formulas, not easily tractable. In particular, these subsets of E can often be viewed as continuous images of Borel sets.

When E is a Polish space (this is the case for $B^s_{p,q}(\mathbb{R}^D)$ when $q < \infty$), such sets are called *analytic*, and we have the following theorem [3].

Theorem 2.6. *Every analytic set in a Polish space is universally measurable.*

When E is not Polish (in our context, when $E = B^s_{p,\infty}(\mathbb{R}^D)$), continuous images of Borel sets need not be universally measurable. Hence, in order to obtain the universal measurability for our specific sets, the definition of an analytic set has to be modified and is more complicated (see § A.1). Once this second definition is adopted, the same result as Theorem 2.6 holds, i.e. analyticity implies universal measurability. The fact that the sets we will meet indeed satisfy this second definition of analytic set is proved in § A.2.

3. Proof of Theorem 1.1: Lebesgue-almost every trace of every function

Recall that $f_a = F_a + G_a$, where F_a and G_a are defined in (13) and (14). Recall also that the wavelet coefficients of f_a are denoted by $d_\lambda(a)$ in its wavelet decomposition (18), while those of f are denoted by c^D_λ.

First, we are going to apply Proposition 2.4. Recalling the definition (15) of the wavelet coefficients of G_a, we need to bound by above the sum

$$\begin{aligned}\sum_{\lambda\in\Lambda_j^d\times 0^d} |d_\lambda(a)|^p &= \sum_{\lambda\in\Lambda_j^d\times 0^d}\left|\sum_{\mathbf{k}'\in\mathbb{Z}_j^{d'},\mathbf{l}'\in L^{d'}} c_{\lambda^D}\prod_{i=1}^{d'}\Psi^{l'_i}_{j,k'_i}(a_i)\right|^p\\ &\leq C\sum_{\lambda^D\in\Lambda_j^D\times\{0^d\times L^{d'}\}} |c_{\lambda^D}|^p\left|\prod_{i=1}^{d'}\Psi^{l'_i}_{j,k'_i}(a_i)\right|^p\\ &\leq C\sum_{\lambda^D\in\Lambda_j^D\times\{0^d\times\{0,1\}^{d'}\}} |c_{\lambda^D}|^p\left|\prod_{i=1}^{d'}\Psi^{l'_i}_{j,k'_i}(a_i)\right|^p \qquad (21)\end{aligned}$$

for some constant C that depends only on Ψ^0 and Ψ^1. Indeed, since Ψ^0 and Ψ^1 are compactly supported, for each a only a finite number (independent of j or a) of terms in the second sum are non zero.

Now, using definition (16) for the wavelet coefficients of F_a, we find that

$$\begin{aligned}\sum_{\lambda\in\Lambda_j^d\times L^d}|d_\lambda(a)|^p &= \sum_{\lambda\in\Lambda_j^d\times L^d}\left|\sum_{\mathbf{k}'\in\mathbb{Z}_j^{d'},\mathbf{l}'\in\{0,1\}^{d'}} c_{\lambda^D}\prod_{i=1}^{d'}\Psi_{j,k_i'}^{l_i'}(a_i)\right|^p\\ &\le C\sum_{\lambda^D\in\Lambda_j^D\times\{L^d\times\{0,1\}^{d'}\}}|c_{\lambda^D}|^p\left|\prod_{i=1}^{d'}\Psi_{j,k_i'}^{l_i'}(a_i)\right|^p \qquad (22)\end{aligned}$$

again for some constant C that depends only on Ψ^0 and Ψ^1.

We now consider the sum of all wavelet coefficients of f_a, and we integrate it over $a\in[0,1]^{d'}$. Using (21) and (22), and recalling that $L^d\cup 0^d=\{0,1\}^d$, we get

$$\int_{[0,1]^{d'}}\sum_{\lambda\in\Lambda_j^d\times\{0,1\}^d}|d_\lambda(a)|^p da\le C\sum_{\lambda^D\in\Lambda_j^D\times\{0,1\}^D}|c_{\lambda^D}|^p\int_{[0,1]^{d'}}\left|\prod_{i=1}^{d'}\Psi_{j,k_i'}^{l_i'}(a_i)\right|^p da.$$

Observe that there is no wavelet coefficient associated with the index $l^D=0^D$ for the function f, hence the sum of λ^D over $\Lambda_j^D\times\{0,1\}^D$ is the same as the sum of λ^D over $\Lambda_j^D\times L^D$.

Since the functions $\Psi_{j,k_i'}^{l_i'}$ are bounded uniformly in j, $\mathbf{k}'$ and $\mathbf{l}'$, and has support width $\le K2^{-j}$,

$$\int_{[0,1]^{d'}}\sum_{\lambda\in\Lambda_j^d\times\{0,1\}^d}|d_\lambda(a)|^p da\le C\sum_{\lambda^D\in\Lambda_j^D\times L^D}|c_{\lambda^D}|^p 2^{-jd'}.$$

Using the definition of Besov norm (9) and that $D=d+d'$, we see that

$$\int_{[0,1]^{d'}}\sum_{\lambda\in\Lambda_j^d\times\{0,1\}^d}|d_\lambda(a)|^p da\le C\|f\|_{B^s_{p,\infty}([0,1]^D)}2^{(d-sp)j}$$

for some other constant C that still depends only on Ψ.

Let us now define

$$\mathcal{A}_j:=\left\{a\in[0,1]^{d'}:\sum_{\lambda\in\Lambda_j^d\times\{0,1\}^d}|d_\lambda(a)|^p>Cj^2\|f\|_{B^s_{p,\infty}([0,1]^D)}2^{(d-sp)j}\right\}$$

By Markov's inequality, it follows that

$$\mathcal{L}_{d'}(\mathcal{A}_j)\le\frac{C\|f\|_{B^s_{p,\infty}([0,1]^D)}2^{(d-sp)j}}{Cj^2\|f\|_{B^s_{p,\infty}([0,1]^D)}2^{(d-sp)j}}=j^{-2}.$$

Thus, applying the Borel-Cantelli lemma, $\mathcal{L}_{d'}(\limsup_j\mathcal{A}_j)=0$.

Finally, by construction, for any $a \in [0,1]^{d'} \setminus \limsup_j \mathcal{A}_j$, there exists j_0 such that $j \geq j_0$ implies

$$\sum_{\lambda \in \Lambda_j^d \times \{0,1\}^d} |d_\lambda(a)|^p \frac{2^{(sp-d)j}}{j^2} \leq C \|f\|_{B^s_{p,\infty}([0,1]^D)}. \tag{23}$$

In particular, recalling the characterization (9), such a trace f_a belongs to $B^{s'}_{p,\infty}([0,1]^d)$, for every $s' < s$. The factor j^2 in (23) explains the loss of a logarithmic order of the Besov regularity mentioned in the introduction. □

4. Proof of Theorem 2.1

4.1. A first property of the wavelet.

Recall the definition (6) of the periodized wavelet G and let us introduce its d'-dimensional version $G_{d'}$ defined as

$$G_{d'} : x \in \mathbb{R}^{d'} \longmapsto G(x_1) \cdot G(x_2) \cdots G(x_{d'}). \tag{24}$$

Proposition 4.1. *If Ψ satisfies $(\mathcal{H}_N)$, then the set*

$$\mathcal{A}_1 := \left\{ a \in [0,1]^{d'} : \exists\, j_a \in \mathbb{N} \text{ such that } \forall j \geq j_a,\ \left|G_{d'}(2^j a)\right| > j^{-2d'} \right\}$$

has full Lebesgue measure.

Remark that Proposition 4.1 holds in fact for any function $G \in C^N(\mathbb{R})$ satisfying assumptions (ii) and (iii) of $(\mathcal{H}_N)$.

Proof. Obviously, if we are able to prove that the set

$$\mathcal{A}_1' := \left\{ a_1 \in [0,1] : \exists\, j_a \in \mathbb{N} \text{ such that } \forall j \geq j_a,\ \left|G(2^j a_1)\right| > j^{-2} \right\}$$

has full Lebesgue measure in $[0,1]$, then Proposition 4.1 will be proved since we have the inclusion $(\mathcal{A}_1')^{d'} \subset \mathcal{A}_1$.

By $(\mathcal{H}_N)$, there is a finite number, say $y_1, y_2, \cdots, y_p$, of zeros of G on the interval $[0,1]$, and G' does not vanish at these real numbers. Let $M = \min(|G'(y_1)|, |G'(y_2)|, \cdots, |G'(y_p)|)$. For each y_i there is a small interval $[y_i - r_i, y_i + r_i]$ around y_i on which $|G(a)| \geq M/2|a - y_i|$.

Let $r = \min(r_i : i = 1, ..., p)$.

Let us denote by m the minimum of G on the compact set $[0,1] \setminus \bigcup_{i=1}^p (y_i - r, y_i + r)$. We now choose an integer n_1 such that $1/n_1 \leq \min(m, r)$.

The above construction guarantees that for every integer $n \geq n_1$, for every $a \notin \bigcup_{i=1}^p [y_i - \frac{2}{M \cdot n}, y_i + \frac{2}{M \cdot n}]$, we have $|G(a)| \geq 1/n$. In other words, $|G(a)| < 1/n$ except on a set of at most Lebesgue measure $\sum_{i=1}^p 2 \frac{2}{M \cdot n} = C/n$, for some constant $C > 0$. This immediately implies that for every j large enough, the set

$$\widetilde{\mathcal{A}}(j) := \{ a \in [0,1] : |G(a)| \leq j^{-2} \}$$

has a Lebesgue measure less than Cj^{-2}.

Remarking the 1-periodicity of G, we deduce that the Lebesgue measure of the set

$$\mathcal{A}'(j) := \{ a \in [0,1] : \left|G(2^j a)\right| \leq j^{-2} \}$$

is also equal Cj^{-2} (the same as that of $\widetilde{\mathcal{A}}(j)$). Obviously,

$$\sum_{j\geq 1} \mathcal{L}(\mathcal{A}'(j)) < +\infty.$$

Thus, applying the Borel-Cantelli lemma to the sets $\mathcal{A}'(j)$, we deduce that the limsup set

$$\bigcap_{J\geq 1} \bigcup_{j\geq J} \mathcal{A}'(j)$$

has zero Lebesgue measure. This set is the complement of the set $\mathcal{A}'_1$, which by deduction is of full Lebesgue measure in $[0,1]$. □

4.2. **Prevalence property of an ancillary set.**

The key result to obtain the prevalence of the singularity spectrum of Theorem 2.1 is the following theorem.

Theorem 4.2. *Suppose that* $0 < s - D/p < \infty$ *and* $0 < q \leq \infty$. *Let* $\alpha \geq 1$ *and let us defined the exponent*

$$H(\alpha) := s - \frac{d}{p} + \frac{d}{\alpha p}. \tag{25}$$

The set

$$\mathcal{F}_\alpha := \Big\{ f \in B^s_{p,q}([0,1]^D) : \exists\, \mathcal{A}(f) \text{ of full Lebesgue measure such that } a \in \mathcal{A}(f) \Longrightarrow \forall x \in \mathcal{X}^\alpha,\ h_{f_a}(x) \leq H(\alpha) \Big\}$$

is prevalent in $B^s_{p,q}([0,1]^D)$.

The proof of Theorem 4.2 is postponed to § 5. We admit it for the moment, and we explain how we conclude once Theorem 4.2 is proved. Our main result, Theorem 2.1, is a direct consequence of Propositions 4.4–4.6 below.

From now on, let $(\alpha_n)_{n\in\mathbb{N}}$ be a dense sequence in $[1,\infty)$ such that $\alpha_0 = 1$. Using the fact that a countable intersection of prevalent (resp. full Lebesgue measure) sets is prevalent (resp. of full Lebesgue measure), it follows immediately that:

Corollary 4.3. *The set*

$$\mathcal{F} := \Big\{ f \in B^s_{p,q}([0,1]^D) : \exists\, \mathcal{A}(f) \text{ of full Lebesgue measure such that } a \in \mathcal{A}(f) \Rightarrow \forall n \in \mathbb{N},\, \forall x \in \mathcal{X}^{\alpha_n},\, h_{f_a}(x) \leq H(\alpha_n) \Big\}$$

is prevalent in $B^s_{p,q}([0,1]^D)$.

4.3. **Prevalent upper bound.**

We first find an upper bound for the singularity spectrum of Lebesgue almost traces of f, for **every** $f \in B^s_{p,q}([0,1]^D)$.

Proposition 4.4. *For every* $f \in B^s_{p,q}([0,1]^D)$, *for almost all* $a \in [0,1]^{d'}$, *we have*

$$\text{for every } h \geq s - d/p, \quad d_{f_a}(h) \leq \min(d, d + (h-s)p). \tag{26}$$

Proof. Let $f \in B^s_{p,q}(\mathbb{R}^D)$. By Theorem 1.1, there is a set $\mathcal{A}(f)$ of full Lebesgue measure in $[0,1]^{d'}$ such that for every $a \in \mathcal{A}(f)$, the trace f_a belongs to $\bigcap_{s-\varepsilon<s} B^{s-\varepsilon}_{p,\infty}(\mathbb{R}^d)$. Then, by Theorem 1.2, for every $h \geq s - d/p$, for every $\varepsilon > 0$,

$$d_{f_a}(h) \leq \min(d, d + (h-s)p + \varepsilon p).$$

Moreover, for every $\varepsilon > 0$, since $f_a \in B^{s-\varepsilon}_{p,\infty}(\mathbb{R}^d)$ for every $a \in \mathcal{A}(f)$, there is no point $x \in [0,1]^d$ such that $h_{f_a}(x) < s - d/p - \varepsilon$.

Letting $\varepsilon > 0$ yields exactly the upper bound (26). □

One can obtain more precise informations for **almost all** $f \in B^s_{p,q}([0,1]^D)$, i.e. on a prevalent set in $f \in B^s_{p,q}([0,1]^D)$.

Proposition 4.5. *For almost all $f \in B^s_{p,q}([0,1]^D)$, for Lebesgue-almost all $a \in [0,1]^{d'}$, for all $x \in [0,1]^d$, $h_{f_a}(x) \leq s$.*

Proof. We apply Corollary 4.3 with $\alpha_n = \alpha_0 = 1$: if f belongs to the prevalent set $\mathcal{F}$, then for any $a \in \mathcal{A}(f)$, for any $x \in \mathcal{X}^{\alpha_0} = \mathcal{X}^1 = [0,1]^d$, $h_{f_a}(x) \leq H(\alpha_0) = s$. □

4.4. **Prevalent lower bound.**

Proposition 4.6. *For almost all $f \in B^s_{p,q}([0,1]^D)$, for almost all $a \in [0,1]^{d'}$, for any $h \in [s - d/p, s]$, $d_{f_a}(h) \geq d + (h-s)p$ and furthermore, $E_{f_a}(s)$ has full Lebesgue measure.*

Proof. Consider a function f in the prevalent set $\mathcal{F}$. Let $h \in (s - d/p, s]$. This exponent can be written

$$h = H(\alpha) = s - \frac{d}{p} + \frac{d}{\alpha p} \tag{27}$$

for some given $\alpha \geq 1$.

Consider a subsequence $(\alpha_{\phi(n)})_{n\in\mathbb{N}}$ of $(\alpha_n)_{n\in\mathbb{N}}$ which is nondecreasing and converges to α (for $\alpha = 1$ this would just be $\phi = 0$).

Let us first assume that $\alpha > 1$, i.e. $H(\alpha) \in (s - d/p, s)$. Remark that $\mathcal{X}^\alpha \subset \bigcap_{n\geq 1} \mathcal{X}^{\alpha_{\phi(n)}}$. Since $f \in \mathcal{F}$, it follows that for all $a \in \mathcal{A}(f)$ and $x \in \mathcal{X}^\alpha$, $h_{f_a}(x) \leq H(\alpha)$. Hence $\mathcal{X}^\alpha \subset \{x : h_{f_a(x)} \leq H(\alpha)\}$.

Recall that Theorem 2.5 provides us with a measure m_α which is supported by $\mathcal{X}^\alpha$, and which gives measure 0 to every set of dimension strictly less than d/α.

Let us introduce the set $\mathcal{Y}^\alpha := \{x : h_{f_a(x)} < H(\alpha)\}$. Clearly,

$$\mathcal{Y}^\alpha = \bigcup_{n\geq 1} \{x : h_{f_a}(x) \leq H(\alpha) - 1/n\}.$$

By (26), each set $\{x : h_{f_a}(x) \leq H(\alpha) - 1/n\}$ has Hausdorff dimension strictly less than d/α. The scaling properties and the σ-additivity of the measure m_α yield that $m_\alpha(\mathcal{Y}^\alpha) = 0$.

Remembering that $m_\alpha(\mathcal{X}^\alpha) > 0$, we have $m_\alpha(\mathcal{X}^\alpha \backslash \mathcal{Y}^\alpha) > 0$. This means equivalently that $m_\alpha(\{x \in \mathcal{X}^\alpha : h_{f_a}(x) = H(\alpha)\}) > 0$. This implies that the set $\{x \in \mathcal{X}^\alpha : h_{f_a}(x) = H(\alpha)\}$ has Hausdorff dimension greater than d/α, and thus

$$d_{f_a}(h) = d_{f_a}(H(\alpha)) = \dim_{\mathcal{H}}\{x : h_{f_a}(x) = H(\alpha)\} \geq d/\alpha = p(h-s) + d,$$

the last equality following from (27).

When $\alpha = 1$, the same reasoning using the d -dimensional Lebesgue measure $\mathcal{L}_d$ instead of m_α yields $E_{f_a}(s) \supset [0,1]^d \backslash \mathcal{Y}^1$ with $\mathcal{L}_d(\mathcal{Y}^1) = 0$. Hence $\mathcal{L}_d(E_{f_a}(s)) = 1$.

Finally, it remains us to treat the case of the smallest exponent $h = s - d/p$. Remembering the definition of $\mathcal{F}$, observe that at any element x of $\mathcal{X}^\infty := \bigcap_{\alpha \geq 1} \mathcal{X}^\alpha = \bigcap_{n \geq 1} \mathcal{X}^{\alpha_n}$, one necessarily has $h_f(x) \leq s - d/p$. Since the converse inequality holds true for any x, we have proved that $\mathcal{X}^\infty \subset E_{f_a}(s - d/p)$. We conclude by noting that $\mathcal{X}^\infty$ is certainly not empty (and uncountable), since it is a dense G_δ set in $\mathbb{R}^d$. □

Theorem 2.1 is now proved, provided that we can establish Theorem 4.2.

5. Proof of Theorem 4.2: prevalence of $\mathcal{F}_\alpha$

We simplify the problem by including the complement of $\mathcal{F}_\alpha$ in a countable union of simpler ancillary sets. Let N be an integer, $\alpha > 1$, $\gamma > H(\alpha)$ and

$$\mathcal{O}_{\gamma,N} := \left\{ f \in B^s_{p,q}([0,1]^D) : \mathcal{L}_{d'}(\mathcal{A}_{\gamma,N}(f)) > 0 \right\}$$

where

$$\mathcal{A}_{\gamma,N}(f) = \left\{ a \in [0,1]^{d'} : \begin{array}{c} \exists x \in \mathcal{X}^\alpha, \ \forall \lambda = (j, \mathbf{k}, l) \in \Lambda^d \times L^d, \\ |c_\lambda(a)| \leq N 2^{-\gamma j}(1 + |2^j x - \mathbf{k}|)^\gamma \end{array} \right\}.$$

Remark that the conditions on the wavelet coefficients that appear in the definition of $\mathcal{A}_{\gamma,N}(f)$ implies that f_a has exponent greater than γ at x.

Recall the definition of $\mathcal{F}_\alpha$

$$\mathcal{F}_\alpha := \Big\{ f \in B^s_{p,q}([0,1]^D) : \exists\, \mathcal{A}(f) \text{ of full Lebesgue measure such that}$$
$$a \in \mathcal{A}(f) \Longrightarrow \forall x \in \mathcal{X}^\alpha, \ h_{f_a}(x) \leq H(\alpha) \Big\}.$$

Proposition 5.1. *For any sequence $(\gamma_n)_{n\in\mathbb{N}}$ strictly decreasing to $H(\alpha)$, we have*

$$B^s_{p,q}([0,1]^D) \backslash \mathcal{F}_\alpha \subset \bigcup_{n,N \in \mathbb{N}} \mathcal{O}_{\gamma_n,N}$$

Proof. We write that $\mathcal{F}_\alpha = \bigcap_{n\geq 1} \mathcal{F}_{\alpha,\gamma_n}$, where for any $\gamma > H(\alpha)$ we put

$$\mathcal{F}_{\alpha,\gamma} = \left\{ f \in B^s_{p,q}([0,1]^D) : \begin{array}{c} \exists\, \mathcal{A}_\gamma(f) \text{ of full Lebesgue measure such that} \\ a \in \mathcal{A}_\gamma(f) \Rightarrow \ \forall x \in \mathcal{X}^\alpha, \ f_a \notin C^\gamma(x) \end{array} \right\}.$$

When $f \notin \mathcal{F}_{\alpha,\gamma}$, the set $\left\{ a \in [0,1]^{d'} : \exists x \in \mathcal{X}^\alpha, f_a \in C^\gamma(x) \right\}$ has positive Lebesgue measure. But by (11) of Proposition 2.3 which gives the characterization of $C^\gamma(x)$ in terms of wavelet coefficients, this last set is included in $\mathcal{A}_{\gamma,N}(f)$, for some $N \geq 1$. Hence $B^s_{p,q}([0,1]^D) \backslash \mathcal{F}_{\alpha,\gamma} \subset \bigcup_{N\in\mathbb{N}^*} \mathcal{O}_{\gamma,N}$ and the conclusion follows. □

To prove Theorem 4.2, it suffices now to show that each set $\mathcal{O}_{\gamma,N}$ is universally measurable (Proposition 5.2) and shy (Proposition 5.7).

From now on we fix $N \in \mathbb{N}^*$, $\alpha > 1$ and $\gamma > H(\alpha)$.

5.1. **Measurability.**

First we deal here only with the case $q < \infty$, that is when $B^s_{p,q}(\mathbb{R}^D)$ is a Polish space. The case $q = \infty$ is proved in Appendix A, Proposition A.1.

Proposition 5.2. *The set $\mathcal{O}_{\gamma,N}$ is universally measurable in $B^s_{p,q}([0,1]^D)$.*

Proof. Let

$$\begin{aligned} \Phi_\lambda(f,a,x) &:= N2^{-\gamma j}(1 + \left|2^j x - k\right|)^\gamma - |c_\lambda(a)| \\ \Phi(f,a,x) &:= \inf_{\lambda \in \Lambda^d \times L^d} \Phi_\lambda(f,a,x) \end{aligned}$$

and

$$\widetilde{\Phi}(f) := \mathcal{L}_{d'}\left(\left\{a \in [0,1]^{d'} : \exists x \in \mathcal{X}^\alpha, \Phi(f,a,x) \geq 0\right\}\right)$$

so that

$$\mathcal{O}_{\gamma,N} = \widetilde{\Phi}^{-1}\big((0,+\infty)\big).$$

To obtain Proposition 5.2, we just have to prove that $\widetilde{\Phi}$ is universally measurable as a map : $B^s_{p,q}([0,1]^D) \longrightarrow \mathbb{R}^+$. For this, let us fix a complete Borel measure μ on $B^s_{p,q}([0,1]^D)$.

First observe that each Φ_λ is continuous on $B^s_{p,q}([0,1]^D) \times [0,1]^{d'} \times [0,1]^d$. This follows from the fact that the dependence of the wavelet coefficients c^D_λ on f is continuous (the Besov topology induced on the space of wavelet coefficients by (9) is stronger than the product topology) and the dependence of $c_\lambda(a)$ on the variables a and c^D_λ is also continuous (from their definitions (15), (16) and (19)). As a countable infimum of continuous functions, Φ is Borel on the Polish space $B^s_{p,q}([0,1]^D) \times [0,1]^{d'} \times [0,1]^d$.

Clearly $\mathcal{X}^\alpha \in \mathcal{B}([0,1]^d)$, so the set

$$\widetilde{\mathcal{T}} := \Phi^{-1}([0,\infty)) \cap \left(B^s_{p,q}([0,1]^D) \times [0,1]^{d'} \times \mathcal{X}^\alpha\right)$$

is also Borel and its projection along the third coordinate

$$\Pi(\widetilde{\mathcal{T}}) := \left\{(f,a) \in B^s_{p,q}([0,1]^D) \times [0,1]^{d'} : \exists x \in \mathcal{X}^\alpha, (f,a,x) \in \widetilde{\mathcal{T}}\right\}$$

is analytic in the space $(B^s_{p,q}([0,1]^D) \times [0,1]^{d'}, \mathcal{B}(B^s_{p,q}([0,1]^D) \times [0,1]^{d'}))$. By the universal measurability Theorem 2.6, $\Pi(\widetilde{\mathcal{T}})$ is then $\mu \otimes \mathcal{L}_{d'}$-measurable.

To conclude, we notice that $\widetilde{\Phi}$ can be written as

$$\widetilde{\Phi} : f \mapsto \int_{[0,1]^{d'}} \mathbf{1}_{\Pi(\widetilde{\mathcal{T}})}(f,a)\, da.$$

Since $\Pi(\widetilde{\mathcal{T}})$ is $\mu \otimes \mathcal{L}_{d'}$-measurable, we can apply Fubini's theorem, so that we conclude that $\widetilde{\Phi}$ is μ-measurable, for any complete Borel measure μ on $B^s_{p,q}([0,1]^D)$. □

5.2. **Probe space.**

In this section $q \in (0,\infty)$, with the obvious modifications when $q=\infty$.

We use the following notation: to each $(j,\mathbf{k}) \in \Lambda^d$ we associate the unique $(J,\mathbf{K})$, $J \in \mathbb{N}$ and $\mathbf{K} \in \mathbb{Z}^d_J \backslash 2\mathbb{Z}^d_J$ satisfying $\mathbf{K}2^{-J} = \mathbf{k}2^{-j}$ ($\mathbf{K}2^{-J}$ is the irreducible version of the dyadic point $\mathbf{k}2^{-j}$). Obviously, with the preceding notations, $J \le j$.

Proposition 5.3. *Let us define, for every* $\lambda^D = (j,(\mathbf{k},\mathbf{k}'),(\mathbf{l},\mathbf{l}')) \in \Lambda^D$

$$e_{\lambda^D} := \begin{cases} j^{-\frac{q+2}{qp}} 2^{(\frac{d}{p}-s)j} 2^{-\frac{d}{p}J} & \text{if } \mathbf{l} \neq 0^d \text{ and } \mathbf{l}' = 1^{d'}, \\ 0 & \text{if } \mathbf{l} = 0^d \text{ or } \mathbf{l}' \neq 1^{d'} . \end{cases} \tag{28}$$

The function $g := \sum_{\lambda^D \in \Lambda^D} e_{\lambda^D} \psi_{\lambda^D}$ *belongs to* $B^s_{p,q}([0,1]^D)$.

Proof. Observe that e_{λ^D} does not depend on $\mathbf{l} \in L^d$. Using the wavelet characterization (9) of $B^s_{p,q}([0,1]^D)$, the proof boils down to studying for all integers $j \ge 1$ the quantity

$$A_j := 2^{j(sp-D)} \sum_{\lambda^D = (j,(\mathbf{k},\mathbf{k}'),(\mathbf{l},\mathbf{l}')):\, (\mathbf{k},\mathbf{k}') \in \mathbb{Z}^D_j,\, (\mathbf{l},\mathbf{l}') \in \{0,1\}^D} |e_{\lambda^D}|^p.$$

By construction, e_{λ^D} is the same for all $\mathbf{k}'$ and all $\mathbf{l}$, and equals zero except when $\mathbf{l}' = 1^{d'}$ and $\mathbf{l} \neq 0^d$. Thus,

$$\begin{aligned} A_j &= (2^d-1) 2^{j(sp-D+d')} \sum_{\mathbf{k} \in \mathbb{Z}^d_j} |e_{\lambda^D}|^p \le 2^{d+j(sp-d)} \sum_{\mathbf{k} \in \mathbb{Z}^d_j} j^{-p\frac{q+2}{qp}} 2^{(d-sp)j} 2^{-dJ} \\ &\le 2^d j^{-\frac{q+2}{q}} \sum_{\mathbf{k} \in \mathbb{Z}^d_j} 2^{-dJ}. \end{aligned}$$

where one should not forget that J depends on $\mathbf{k}$. For a given integer $1 \le J \le j$, the number of multi-integers $\mathbf{k} \in \mathbb{Z}^d_j$ such that its irreducible version can be written $\mathbf{K}2^{-J}$ (for some $\mathbf{K}$) is exactly $2^{d(J-1)}$. Hence

$$A_j \le 2^d j^{-\frac{q+2}{q}} \sum_{J=1}^{j} 2^{d(J-1)-dJ} = j^{-\frac{2}{q}}$$

which is an l^q sequence. □

Remark 5.4. *Although we did not prove it here, the singularity spectrum of* g *(and of the functions* $g^{(i)}$ *below) can be explicitly computed: for every* $h \in [s-d/p, s]$, $d_g(h) = ph - ps + D$, *and* $d_g(h) = -\infty$ *else. It is noticeable that* g *does not enjoy the generic spectrum in* $B^s_{p,q}([0,1]^D)$ *(the generic spectrum has the same formula but the range of* h *is* $[s-D/p, s]$, *not* $[s-d/p, s]$*). Nevertheless its traces will be shown to have the typical spectrum in* $B^s_{p,q}([0,1]^d)$.

Let $J_0 \ge 1$ to be fixed later and $d_1 := 2^{dJ_0}$. For each d-dimensional dyadic cube $\lambda \in \Lambda^d$ at scale j, we enumerate in an arbitrary fashion $\lambda^{(1)}, \dots, \lambda^{(d_1)}$ its d_1 sub-cubes at scale $j + J_0$.

Definition 4. *We set the* probe space $\mathcal{P}$ *to be the* d_1*-dimensional subspace of* $B^s_{p,q}([0,1]^D)$ *spanned by the functions* $g^{(i)}$*, whose wavelet coefficients* $e^{(i)}_{\lambda^D}$ *are defined in the following way: for each* $\lambda^D = (j,(\mathbf{k},\mathbf{k}'),(\mathbf{l},\mathbf{l}')) \in \Lambda^D$*, let* $\lambda := (j,\mathbf{k},\mathbf{l})$ *and*

$$e^{(i)}_{\lambda^D} = \begin{cases} e_{\widetilde{\lambda}^D} & \text{if } \lambda = \widetilde{\lambda}^{(i)} \text{ for some } \widetilde{\lambda}^D := (j-J_0,(\tilde{\mathbf{k}},\tilde{\mathbf{k}}'),(\tilde{\mathbf{l}},\tilde{\mathbf{l}}')) \\ 0 & \text{else.} \end{cases} \tag{29}$$

In the definition above, $\widetilde{\lambda}^{(i)}$ is the sub-cube associated with $\widetilde{\lambda} = (j - J_0, \tilde{\mathbf{k}}, \tilde{\mathbf{l}})$ (which is the restriction to $[0,1]^d$ of $\widetilde{\lambda}^D$).

In particular, recalling (28), as soon as $\mathbf{l}' \neq 1^{d'}$, $e^{(i)}_{\lambda^D} = 0$, and this coefficient is the same for all $\mathbf{k}' \in \mathbb{Z}^{d'}_j$. By the same proof as Proposition 5.3, each $g^{(i)}$ also belongs to $B^s_{p,q}([0,1]^D)$.

Heuristically, the wavelet coefficients of g at generation j are dispatched in wavelet coefficients at generation $j+J_0$ for the functions $g^{(i)}$, the distribution being organized so that for any cube λ^D, there is only one $g^{(i)}$ such that $e^{(i)}_{\lambda^D} \neq 0$.

Let us now consider their traces $g^{(i)}_a$ on the affine subspace $\mathcal{H}_a$.

Lemma 5.5. *For every* $i \in \{1,\cdots,d_1\}$*, for every* $j \geq 1$*, for every* $\lambda = (j,\mathbf{k},\mathbf{l})$ *with* $\mathbf{k} \in \mathbb{Z}^d_j$ *and* $\mathbf{l} \neq 0^d$*, we have the formula*

$$e^{(i)}_{\lambda}(a) = e^{(i)}_{(j,(\mathbf{k},1^{d'}),(1^d,1^{d'}))} G_{d'}(2^j a) \tag{30}$$

where $G_{d'}$ *was defined in* (24).

Moreover, if $\mathbf{l} = 0^d$*, then* $e^{(i)}_{\lambda}(a) = 0$.

Proof. Following (15) and (16), the wavelet coefficients of these traces are: for all $j \geq 1$, for all $\lambda = (j,\mathbf{k},\mathbf{l}) \in \{j\} \times \mathbb{Z}^d_j \times \{0,1\}^d$,

$$\text{if } \mathbf{l} = 0^d, \quad e^{(i)}_{\lambda}(a) := \sum_{\substack{\lambda^D = (j,(\mathbf{k},\mathbf{k}'),(0^d,\mathbf{l}')): \\ \mathbf{k}' \in \mathbb{Z}^{d'}_j,\, \mathbf{l}' \in L^{d'}}} e^{(i)}_{\lambda^D} \prod_{i=1}^{d'} \Psi^{l'_i}_{j,k'_i}(a_i). \tag{31}$$

$$\text{if } \mathbf{l} \in L^d, \quad e^{(i)}_{\lambda}(a) := \sum_{\substack{\lambda^D = (j,(\mathbf{k},\mathbf{k}'),(\mathbf{l},\mathbf{l}')): \\ \mathbf{k}' \in \mathbb{Z}^{d'}_j,\, \mathbf{l}' \in \{0,1\}^{d'}}} e^{(i)}_{\lambda^D} \prod_{i=1}^{d'} \Psi^{l'_i}_{j,k'_i}(a_i). \tag{32}$$

By definition of $e^{(i)}_{\lambda^D}$, the coefficients (31) are all zero. Now, remember that by construction $e^{(i)}_{\lambda^D}$ does not depend on $\mathbf{k}'$, nor on $\mathbf{l} \in L^d$, and that they all have the same values as one of them, say the one with $\mathbf{l} = 1^d$. Thus, as soon as $\mathbf{l} \neq 0^d$, formula (32) can be simplified into

$$\begin{aligned} e^{(i)}_\lambda(a) &= e^{(i)}_{(j,\mathbf{k},1^d)}(a) = \sum_{\lambda^D=(j,(\mathbf{k},\mathbf{k}'),(1^d,1^{d'}) :\ \mathbf{k}'\in\mathbb{Z}^{d'}_j} e^{(i)}_{\lambda^D} \prod_{i=1}^{d'} \Psi^1_{j,k'_i}(a_i) \\ &= e^{(i)}_{(j,(\mathbf{k},1^{d'}),(1^d,1^{d'}))} \sum_{\mathbf{k}'\in\mathbb{Z}^{d'}_j} \prod_{i=1}^{d'} \Psi^1(2^j a_i - k'_i). \end{aligned}$$

Since Ψ^1 has compact support, for a given $a \in (0,1)^{d'}$, when j is large enough, we have

$$\sum_{\mathbf{k}'\in\mathbb{Z}^{d'}_j} \prod_{i=1}^{d'} \Psi^1(2^j a_i - k'_i) = \sum_{\mathbf{k}'\in\mathbb{Z}^{d'}} \prod_{i=1}^{d'} \Psi^1(2^j a_i - k'_i) = G_{d'}(2^j a).$$

This yields (30). □

Let $x \in \mathcal{X}^\alpha$ not a dyadic element of $[0,1]^d$, and consider the irreducible sequence $(J_n, \mathbf{K}_n)$ associated to x as in (20), i.e.

$$|x - \mathbf{K}_n 2^{-J_n}| \leq 2^{-\alpha J_n}.$$

Let $a \in \mathcal{A}_1$ and let j_a be the associated integer constructed in Proposition 4.1. Let n be such that $j_n := [\alpha J_n] \geq j_a$ and such that (30) holds. Let us denote by $\lambda_n := (j_n, \mathbf{k}_n, \mathbf{l})$ the unique dyadic node (unique in the sense that $\mathbf{l}$ varies in L^d) such that $\mathbf{K}_n 2^{-J_n} = \mathbf{k}_n 2^{-j_n}$. With each λ_n can be associated its sub-cubes $\lambda_n^{(i)}$, $i \in \{1, \cdots, d_1\}$.

Lemma 5.6. *For all $1 \leq i \leq d_1$, $\lambda_n^{(i)}$ lies within the cone of influence of width 2^{J_0+2} above x, and*

$$\left|e^{(i)}_{\lambda_n^{(i)}}(a)\right| \geq C j_n^{-(2d'+\frac{q+2}{qp})} 2^{-H(\alpha)j_n} \tag{33}$$

the constant C depending only on J_0.

Proof. Remark that $\lambda_n^{(i)}$ can be written $\lambda_n^{(i)} = (j_n + J_0, \mathbf{k}_n^{(i)}, \mathbf{l})$ for some integer $\mathbf{k}_n^{(i)} \in \mathbb{Z}^d_{j_n+J_0}$. By construction,

$$|\mathbf{k}_n^{(i)} 2^{-(j_n+J_0)} - \mathbf{K}_n 2^{-J_n}| = |\mathbf{k}_n^{(i)} 2^{-(j_n+J_0)} - \mathbf{k}_n 2^{-j_n}| \leq 2^{-j_n}.$$

Using (20) we deduce that

$$\begin{aligned} \left|x - \mathbf{k}_n^{(i)} 2^{-(j_n+J_0)}\right| &\leq \left|x - \mathbf{K}_n 2^{-J_n}\right| + \left|\mathbf{k}_n^{(i)} 2^{-(j_n+J_0)} - \mathbf{K}_n 2^{-J_n}\right| \\ &\leq 2^{-\alpha J_n} + 2^{-j_n} \leq 3 2^{-j_n} \leq (2^{J_0+2}) 2^{-(j_n+J_0)}. \end{aligned}$$

This shows the first part of Lemma 5.6.

Recall now Proposition 4.1. The fact that $a \in \mathcal{A}_1$ guarantees that $|G_{d'}(2^j a)| \geq j^{-2d'}$ as soon as $j \geq j_a$. Combining this with (30), we get

$$\left|e^{(i)}_{(\lambda_n)^{(i)}}(a)\right| \geq (j_n + J_0)^{-2d'} \left|e^{(i)}_{(j_n+J_0,(\mathbf{k}_n^{(i)},1^{d'}),(1^d,1^{d'}))}\right|.$$

Remembering now how we chose the coefficients of $g^{(i)}$ in (29), we see that

$$\begin{aligned}\left|e^{(i)}_{(\lambda_n)^{(i)}}(a)\right| &\geq (j_n+J_0)^{-2d'}\cdot\left|e_{(j_n,(\mathbf{k}_n,1^{d'}),(1^d,1^{d'}))}\right|\\ &\geq (j_n+J_0)^{-2d'}\cdot(j_n)^{-\frac{q+2}{qp}}2^{(\frac{d}{p}-s)j_n-\frac{d}{p}J_n}\\ &\geq Cj_n^{-(2d'+\frac{q+2}{qp})}2^{-H(\alpha)j_n}\end{aligned}$$

where we used that $\alpha J_n \leq j_n+1$. □

5.3. **Shyness of $\mathcal{O}_{\gamma,N}$.**

Recall that $\gamma > H(\alpha)$.

Take an arbitrary $f\in B^s_{p,q}([0,1]^D)$ with wavelet coefficients c^D_λ, and for each $\beta\in\mathbb{R}^{d_1}$ define

$$f^\beta := f+\sum_{i=1}^{d_1}\beta_i g^{(i)}.$$

As usual now, f^β_a will denote its trace at level $x'=a$ and $c^\beta_\lambda(a)$ the wavelet coefficients of that trace. Now we choose J_0 large enough so that

$$d-d_1(\gamma-H(\alpha))<0 \tag{34}$$

Our goal is to prove:

Proposition 5.7. *For any $f\in B^s_{p,q}([0,1]^D)$, the set $\{\beta\in\mathbb{R}^{d_1}: f^\beta\in\mathcal{O}_{\gamma,N}\}$ has d_1-dimensional Lebesgue measure $\mathcal{L}_{d_1}$ equal to 0.*

This will show that $\mathcal{O}_{\gamma,N}$ is shy. Let us quickly explain this fact.

Let us denote by μ the measure $\mathcal{L}_{d_1}$ carried by $\mathcal{P}$. Assume that Proposition 5.7 holds true, and fix any $f\in B^s_{p,q}([0,1]^D)$. For μ-almost $F\in\mathcal{P}$, we know that $f+F\notin\mathcal{O}_{\gamma,N}$. Hence the set $\{f+\mathcal{O}_{\gamma,N}\}\cap\mathcal{P}$ has a μ-measure equal to 0, i.e.

$$\mu(\{f+\mathcal{O}_{\gamma,N}\})=0.$$

Since this is true for any $f\in B^s_{p,q}([0,1]^D)$, by Definition 1, the set $\mathcal{O}_{\gamma,N}$ is shy.

Before that, two intermediary lemmas are necessary. Let us introduce

$$\mathcal{B}_a := \left\{\beta\in\mathbb{R}^{d_1}: \begin{array}{c}\exists\, x_\beta\in\mathcal{X}^\alpha,\ \forall\,\lambda\in\Lambda^d,\\ \left|c^\beta_\lambda(a)\right|\leq N2^{-\gamma j}(1+\left|2^j x_\beta-\mathbf{k}\right|)^\gamma\end{array}\right\}$$

Lemma 5.8. *The application $(a,\beta)\mapsto \mathbf{1}_{\mathcal{B}_a}(\beta)$ is Lebesgue-measurable as an application from $[0,1]^{d'}\times\mathbb{R}^{d_1}$ to $\mathbb{R}$.*

Proof. Let $\phi : (a,\beta,x)\mapsto \inf_{\lambda\in\Lambda^d} N2^{-\gamma j}(1+\left|2^j x-\mathbf{k}\right|)^\gamma - \left|c^\beta_\lambda(a)\right|$. By an argument similar to the one used in proving Proposition 5.2, ϕ is Borel on $[0,1]^{d'}\times\mathbb{R}^{d_1}\times[0,1]^d$. Remark then that $\mathbf{1}_{\mathcal{B}_a}(\beta)$ can be written as $\mathbf{1}_{\mathcal{B}_a}(\beta)=\mathbf{1}_{\mathcal{G}}(a,\beta)$, where

$$\mathcal{G} := \left\{(a,\beta)\in[0,1]^{d'}\times\mathbb{R}^{d_1}: \exists x\in\mathcal{X}^\alpha, \phi(a,\beta,x)\geq 0\right\}.$$

This set can be written as

$$\pi\left(\phi^{-1}([0,\infty))\bigcap\left([0,1]^{d'}\times\mathbb{R}^{d_1}\times\mathcal{X}^{\alpha}\right)\right), \tag{35}$$

where $\pi(a,\beta,x)=(a,\beta)$ is the (continuous) canonical projection on the two first coordinates. Since the set between brackets in (35) is clearly a Borel set, $\mathcal{G}$ is analytic and in particular, by Theorem 2.6, it is Lebesgue-measurable. By a Fubini argument, we deduce Lemma 5.8. □

Lemma 5.9. *For each $a\in\mathcal{A}_1$, the set $\mathcal{B}_a$ has Lebesgue measure 0.*

Proof. For any $\lambda_0:=(j_0,\mathbf{k}_0,\mathbf{l}_0)\in\Lambda^d$ we put

$$\mathcal{B}_{a,\lambda_0}:=\left\{\beta\in\mathbb{R}^{d_1}:\begin{array}{c}\exists\, x_\beta\in B(\mathbf{k}_0 2^{-j_0},2^{-\alpha j_0}),\forall\lambda\in\Lambda^d,\\ \left|c^{\beta}_{\lambda}(a)\right|\le N2^{-\gamma j}(1+\left|2^j x_\beta-\mathbf{k}\right|)^{\gamma}\end{array}\right\}$$

so that

$$\mathcal{B}_a=\limsup_{j_0\to\infty}\bigcup_{k_0\in\mathbb{Z}^d_{j_0}}\mathcal{B}_{a,\lambda_0}=\bigcap_{j\ge1}\bigcup_{j_0\ge j}\bigcup_{\mathbf{k}_0\in\mathbb{Z}^d_{j_0}}\mathcal{B}_{a,\lambda_0}$$

We want to show that $\mathcal{L}_{d_1}(\mathcal{B}_a)=0$ by bounding by above each $\mathcal{L}_{d_1}(\mathcal{B}_{a,\lambda_0})$. Suppose that β and $\widetilde{\beta}$ both belong to some $\mathcal{B}_{a,\lambda_0}$, where j_0 is large enough so that $j_1:=\lfloor\alpha j_0\rfloor\ge j_a$ (cf. Proposition 4.1).

Applying Lemma 5.6, there exists $\lambda_1=(j_1,\mathbf{k}_1,\mathbf{l}_1)$ such that for all $1\le i\le d_1$,

(i) $\lambda_1^{(i)}$ is in the cone of influence of width 2^{J_0+1} above x_β and $x_{\widetilde{\beta}}$,

(ii) $\left|e^{(i)}_{\lambda_1^{(i)}}(a)\right|\ge Cj_1^{-(q_{d'}+\frac{q+2}{qp})}2^{-H(\alpha)j_1}$,

From (i) we deduce that

$$\left|c^{\beta}_{\lambda_1^{(i)}}(a)\right|=\left|c_{\lambda_1^{(i)}}(a)+\sum_{i=1}^{d_1}\beta_i e^{(i)}_{\lambda_1^{(i)}}(a)\right|\le\frac{C}{2}2^{-\gamma j_1}$$

(C depending on N,J_0,γ) and the same for $\widetilde{\beta}$. On the other hand, by construction of the functions $g^{(i)}$, we have $e^{(i')}_{\lambda_1^{(i)}}(a)=0$ for any $i\neq i'$. Thus

$$\begin{aligned}\left|(\beta_i-\widetilde{\beta}_i)e^{i}_{\lambda_1^{(i)}}(a)\right|&=\left|\left(c_{\lambda_1^{(i)}}(a)+\sum_{i=1}^{d_1}\beta_i e^{i}_{\lambda_1^{(i)}}(a)\right)\right.\\&\qquad\left.-\left(c_{\lambda_1^{(i)}}(a)+\sum_{i=1}^{d_1}\widetilde{\beta}_i e^{i}_{\lambda_1^{(i)}}(a)\right)\right|\\&\le\left|c_{\lambda_1^{(i)}}(a)+\sum_{i=1}^{d_1}\beta_i e^{i}_{\lambda_1^{(i)}}(a)\right|+\left|c_{\lambda_1^{(i)}}(a)+\sum_{i=1}^{d_1}\widetilde{\beta}_i e^{i}_{\lambda_1^{(i)}}(a)\right|\\&\le C2^{-\gamma j_1}.\end{aligned}$$

Recall that $\gamma>H(\alpha)$. Combining this with (ii), we deduce that

$$\left|\beta_i-\widetilde{\beta}_i\right|\le C2^{-(\gamma-H(\alpha))j_1}j_1^{q_{d'}+\frac{q+2}{qp}}$$

hence

$$\mathcal{L}_{d_1}(\mathcal{B}_{a,\lambda_0}) \le C2^{-d_1(\gamma-H(\alpha))j_1} j_1^{d_1(q_{d'}+\frac{q+2}{qp})}$$

Summing over all 2^{dj_0} nodes λ_0 at scale j_0 we conclude that

$$\begin{aligned}\mathcal{L}_{d_1}\left(\bigcup_{k_0\in\mathbb{Z}^d_{j_0}} \mathcal{B}_{a,\lambda_0}\right) &\le C2^{dj_0-d_1(\gamma-H(\alpha))j_1} j_1^{(q_{d'}+\frac{q+2}{qp})d_1}\\ &\le C2^{j_0(d-d_1\alpha(\gamma-H(\alpha)))}(\alpha j_0)^{(q_{d'}+\frac{q+2}{qp})d_1}\end{aligned}$$

whose series converges because of our choice (34) for J_0. The Borel-Cantelli lemma implies that $\mathcal{L}_{d_1}(\mathcal{B}_a) = 0$. □

Proof of Proposition 5.7. We can rewrite the result of Lemma 5.9 as

$$\int_{[0,1]^{d'}} \int_{\mathbb{R}^{d_1}} \mathbf{1}_{\mathcal{B}_a}(\beta)\, d\beta\, da = 0$$

Applying Fubini's theorem (measurability being guaranteed by Lemma 5.8),

$$\int_{\mathbb{R}^{d_1}} \int_{[0,1]^{d'}} \mathbf{1}_{\mathcal{B}_a}(\beta)\, da\, d\beta = 0$$

In other words, for almost all $\beta \in \mathbb{R}^{d_1}$, there exists a set $\mathcal{A}_\beta(f)$ of full Lebesgue measure in $[0,1]^{d'}$ such that $a \in \mathcal{A}_\beta(f)$ implies $\beta \notin \mathcal{B}_a$. This in turn implies $f^\beta \notin \mathcal{O}_{\gamma,N}$ and the announced result follows by complementarity in $\mathbb{R}^{d_1}$. □

Our proof is now complete.

Appendix A. Proof of Proposition 5.2, case $q = \infty$

The only serious difference is due to the fact that $B^s_{p,\infty}([0,1]^D)$ is no longer separable, so the argument for universal measurability of the ancillary set

$$\mathcal{O}_{\gamma,N} := \left\{ f \in B^s_{p,\infty}([0,1]^D) : \mathcal{L}_{d'}(\mathcal{A}_{\gamma,N}(f)) > 0 \right\}$$

has to use a different definition of analyticity.

A.1. **Analytic sets in non-Polish spaces.**

Analytic sets were previously defined in Polish spaces as continuous images of Borel sets: this cannot apply to $B^s_{p,\infty}([0,1]^D)$. However we can use the following more general definition, adapted from [3], for any Hausdorff topological space X endowed with its Borel σ-algebra $\mathcal{B}(X)$. First, for a compact topological space K we write $\mathcal{K}$ the collection of its closed subsets and $(\mathcal{B}(X)\times\mathcal{K})_{\sigma\delta}$ the collection of countable intersection of countable unions of sets that are Cartesian products of a Borel set in X and a closed set in K and $\pi : X \times K \to X$ is the canonical projection map $\pi(x, y) = x$.

Definition 5. *A set $A \subset X$ is said to be* analytic *if there exists a compact space K and $S \in (\mathcal{B}(X) \times \mathcal{K})_{\sigma\delta}$ such that*

$$A = \pi(S).$$

It is easy to check that this definition coincides with the previous one when X is Polish. Furthermore, in this framework, Theorem 2.6 (based on Choquet's capacitability theorem) now holds in any Hausdorff topological space (see [3]).

A.2. **Measurability.**

Proposition A.1. *The set $\mathcal{O}_{\gamma,N}$ is universally measurable in $B^s_{p,\infty}([0,1]^D)$.*

Proof. We use the same notations Φ_λ, Φ, $\widetilde{\mathcal{T}}$ and Π as in the proof of Proposition 5.2. We will show that the set $\Pi(\widetilde{\mathcal{T}})$ is analytic in the sense of Definition 5, with $X := B^s_{p,\infty}([0,1]^D) \times [0,1]^{d'}$ and $K := [0,1]^d$. For short let us put $\Delta_j := \mathbb{Z}_j^{d'} \times \{0,1\}^{d'}$. Given $n \in \mathbb{N}$, $m \in \mathbb{Z}^{\Delta_j}$ and $m' \in \mathbb{Z}^{d'}$ we write

$$Q(n,m) := 2^{-n} \prod_{(k',l') \in \Delta_j} [m_{k',l'}, m_{k',l'}+1]$$

$$Q'(n,m') := 2^{-n} \prod_{1 \le i \le d'} [m'_i, m'_i+1]$$

$$Q(n,m,m') := Q(n,m) \times Q'(n,m')$$

Having fixed $\lambda = (j, \mathbf{k}, \mathbf{l}) \in \Lambda^d \times L^d$ and considering $\lambda^D = (j, (\mathbf{k}, \mathbf{k}'), (\mathbf{l}, \mathbf{l}'))$, any $f \in B^s_{p,\infty}([0,1]^D)$ induces a map $s_\lambda : (\mathbf{k}', \mathbf{l}') \mapsto c_{\lambda^D}$ that we identify to an element of $\mathbb{R}^{\Delta_j}$. Then we define

$$\Theta(s_\lambda, a, x) := N 2^{-\gamma j} (1 + |2^j x - \mathbf{k}|)^\gamma - |c_\lambda(a)|$$

as well as

$$X_\lambda(n,m,m') := \left\{ x \in [0,1]^d : \sup_{(s_\lambda, a) \in Q(n,m,m')} \Theta(s_\lambda, a, x) \ge 0 \right\}$$

The dependency of $c_\lambda(a)$ on s_λ and a is given in (15), (16) and (19) and it is continuous. So is the function Θ. Since $Q(n,m,m')$ is compact, it follows that $X_\lambda(n,m,m')$ is closed (Lemma A.2). Furthermore, if we put

$$F_\lambda(n,m) := \left\{ f \in B^s_{p,\infty}([0,1]^D) : s_\lambda \in Q(n,m) \right\}$$

then it is clear by continuity of Φ_λ that

$$\Phi_\lambda^{-1}([0,\infty)) = \bigcap_{n \in \mathbb{N}} \bigcup_{(m,m') \in \mathbb{Z}^{\Delta_j} \times \mathbb{Z}_j^{d'}} F_\lambda(m,n) \times Q'(m',n) \times X_\lambda(m,m',n)$$

This proves that $\Phi_\lambda^{-1}([0,\infty)) \in (\mathcal{B}(X) \times \mathcal{K})_{\sigma\delta}$. We deduce that $\Phi^{-1}([0,\infty)) = \bigcap_\lambda \Phi_\lambda^{-1}([0,\infty))$ and the set $\widetilde{\mathcal{T}}$ belong to $(\mathcal{B}(X) \times \mathcal{K})_{\sigma\delta}$ as well because $B^s_{p,\infty}([0,1]^D) \times [0,1]^{d'} \times \mathcal{X}^\alpha$ is obviously in $(\mathcal{B}(X) \times \mathcal{K})_{\sigma\delta}$. Its projection $\Pi(\widetilde{\mathcal{T}})$ is thus analytic and we conclude in the same way as for Proposition 5.2. $\square$

Lemma A.2. *Let A and B be topological spaces, A compact and B locally compact. If f is continuous: $A \times B \to \mathbb{R}$, then $f_s : b \mapsto \sup_{a \in A} f(a,b)$ is continuous on B.*

Proof. Recall that a Hausdorff space-valued function defined on a compact set is continuous if and only if its graph is compact. Continuity being a local property, we can suppose without loss of generality that B is also compact. The graph Γ of f is then compact and so is its image by the projection $\varpi : (a, b, y) \mapsto (b, y)$. As a supremum of continuous functions, f_s is lower semi-continuous, so its epigraph E is closed. But the graph of f_s is precisely $E \cap \varpi(\Gamma)$, so it is compact; it follows that f_s is continuous. □

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